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A DYNAMICAL APPROACH TO THE NONLINEAR ANALYSIS OF THE STATE STABILITY OF ELASTIC SYSTEMS

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ABSTRACT
We present methods of statical and dynamical stability analysis of elastic systems in the postcritical area, including both linear and nonlinear analysis, as well as the discussion on how it can be used based on the study of oscillations with large amplitudes and phase trajectories. Analysis is performed on single and double pendulum loaded with axial force and supported by angular springs. Linear analysis is used to determine critical points, and nonlinear analysis is used to determine behaviour of the system after the loss of stability. The results of calculation of buckling eigenmodes are used to simplify nonlinear analysis and to obtain approximate analytic equations of postbuckling trajectory and phase trajectories, as well as frequencies of small vibrations around current secondary equilibrium position. Increase of the amplitude of vibrations around the primary equilibrium position changes frequencies of these vibrations. Equations presented in this paper show also that this well known fact can be used in the analysis of postcritical behaviour of observed systems.
Keywords: dynamical stability, nonlinear analysis, phase trajectories, large displacements.

1. INTRODUCTION
An analysis of state stability as well as boundness of motion is problem which is commonly present in everyday practice. Despite stability is dynamical problem, statical methods of analysis are often used. Reason is in the fact that dynamical analysis requires introduction of mass distribution and damping, and also one parameter more – time. Statical analysis is complex and time consuming, and consideration of equation of motion make this analysis more complex, especially numerical analysis, where inadequate time step may caused instability of a solution. Present research shows that dynamical analysis is appropriate for different systems, while statical analysis is limited to conservative ones. Linear stability analysis is based on derivation of equations of small deformations or vibrations, and analysis of their eigenvalues [1], [4]. According to Liapunov stability theory, analysed system is stable if real part of eigenvalue is positive, [1], [5]. Practical realisation of eigenvalue determination is based on the Routh-Hourwitz criterion [1], or, in analysis of systems with more d.o.f., on the solution of algebraic eigenvalue problem. Nonlinear stability analysis of specific state is also based on linearization of nonlinear equilibrium equation (or equation of motion) and analysis of signs of the eigenvalues (stable node, unstable node, …) [4], or representation of equilibrium path as power series and checking signs of add and even members of the series [2], [3], [6]. Problem, which appears, is identification of stationary points which is connected with solution of sets of nonlinear equations. One of goals of this paper is overview and consideration of existing linear and nonlinear methods for stability analysis, as well as the discussion on how it can be used based on the study of oscillations with large amplitudes and phase trajectories. Analysis is performed on single and double pendulum loaded with axial force and supported by nonlinear springs, as two examples of a conservative system with a known stable symmetric bifurcation of the equilibrium. Nonlinear constitutive equations for these systems are derived. Linear analysis is used to determine critical points, and nonlinear analysis is used to determine behaviour of the system after the loss of stability of

the primary equilibrium position (i.e. after the buckling occurred). Using numerical approach to solve nonlinear constitutive equations it is shown that deformed shape of the system can be approximated in a very wide range by the first buckling eigenmode, itself obtained by linear analysis. This result is used to simplify nonlinear analysis and to obtain approximate analytic equations of postbuckling trajectory and phase trajectories, as well as frequencies of small vibrations around current secondary equilibrium position. Entire process of stability loss is connected with large amplitudes or displacements, and consequently leads to change of frequencies of vibrations [4], [5]. In this paper is also presented possibility of connections of these values with characteristics of bifurcation of stability state, what may be used for elimination of the problem of solution of nonlinear sets of equations in order to identify stationary points. Changes of frequencies are determined for systems with one d.o.f using analytical methods.

2. IDENTIFICATION OF CRITICAL POINTS IN LINEAR ANALYSIS

Linear stability analysis is based on determination of stationary (critical) points of potential energy function, equilibrium path or equation of motion. It is determining value of the applied load for which is possible existence of other equilibrium position except initial, in case of statical analysis, or boundness of motion in case of dynamical analysis. Lets consider stability of the system with one d.o.f., presented on the Fig.1-a. Equation of motion around vertical equilibrium position is

$$ml^2\ddot{\theta} + k\theta = Pl\sin\theta\,, \qquad (1)$$

where m is mass, l is length, and k is spring stiffness.

In linear static case, neglecting inertial and linearizing nonlinear term, equation (1) posses form of homogenous equation which has nontrivial solution only if $P = P_{cr} = k/l$. In this case displacement θ may receive any value. Left side of the equation (1) is equation of harmonic oscillations with angular frequency $\omega_0^2 = (k - Pl)/m$. These oscillations remains bounded until holds $P \le k/l$. Because of neglecting higher order terms in equations, it is not possible predict behaviour for $P > P_{cr}$.

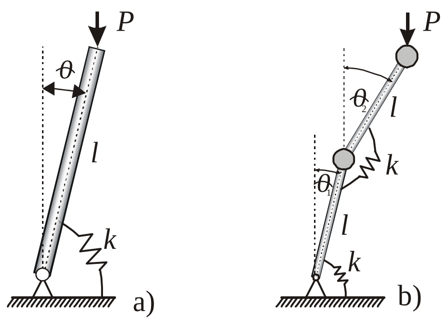

Figure 1. One d.o.f. system a) and two d.o.f. system b) with stable symetric bifurcation .

In case of system with two d.o.f. presented on Fig. 1-b, we form approximate equilibrium equation

$$ml^2\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{Bmatrix} + \begin{bmatrix}2k-Pl & -k\\ -k & k-Pl\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix} = -\frac{Pl}{6}\begin{bmatrix}\theta_1^2 & 0\\ 0 & \theta_2^2\end{bmatrix}\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}. \qquad (2)$$

In statical case, neglecting acceleration and right side terms, equation (2) transforms itself in algebraic eigenvalue problem. Its trivial solution is $\{\theta_1\ \theta_2\}^T = \{0\ 0\}^T$. Existence of the other solutions is possible in the case of vanishing of system determinant. Using it, we obtain forces for which it is possible (critical forces) and corresponding buckling modes

$$P_{1,2} = (3 \mp \sqrt{5})k/(2l)\,,\ \theta_2 = (1 \pm \sqrt{5})\theta_1/2\,. \qquad (3)$$

In analysis of small oscillations, considering small amplitudes in shape of $\{\theta_1\ \theta_2\}^T\sin\omega t$, we derive from (2) algebraic eigenvalue problem which solutions are eigenfrequencies

$$\omega_{1,2}^2 = (6k - 3Pl \pm \sqrt{5P^2l^2 - 24kPl + 32k^2})/(2ml^2)\,, \qquad (4)$$

which also shows boundness of the motion only if $P \le P_1$.

3. NONLINEAR ANALYSIS OF POSTCRITICAL BEHAVIOUR

Nonlinear equation (1) could be, in statical case, expanded in power series as

$$P/P_{cr} = \theta/\sin\theta = 1 + 1/6\theta^2 + 7/360\theta^4 + \cdots. \qquad (5)$$

Positive coefficients of even powers and zero of odd powers shows on stable raising path.

In case of statical analysis of two d.o.f. system, deriving of equation of postcritical path requires formulation of total potential energy and calculation of derivatives $\mathrm{d}P/\mathrm{d}\theta_1$. Postcritical path for system on Fig.1-b, arising from critical point could be written as

$$P/P_{cr} = 1 + (3+\sqrt{5})/(5+\sqrt{5})\theta_1^2 + \cdots . \qquad (6)$$

On Fig. 2 postcritical path (6) is plotted (a) with path obtained by numerical solving using Newton method (b). Considering deformed shape as $\theta_1\{1\ \theta_2/\theta_1\}^T$, where θ_2/θ_1 corresponds to first buckling mode (3), equation (2) may be transformed in one equation (Fig.2-c). It is seen that this mode may be used for approximation of deformed shape in a wide range.

Dynamic approach based on linearization of equilibrium equation in critical points leads to conclusion stated in previous section. Checking eigenvalues of postcritical equilibrium positions may be done if we iteratively solve (2) and calculate eigenvalues in every new equilibrium position.

In the next sections are presented some possibilities of dynamical analysis which may be used to overcame difficulties stated above.

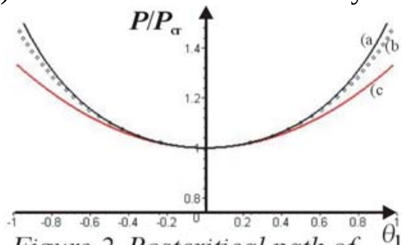

Figure 2. Postcritical path of systems on Figure 1-b.

4. EIGENFREQUENCIES OF POSTCRITICAL MOTION

Lets analyse again system on Fig. 1-a, taking in account nonlinear term. Lets consider that bar under $P > P_{cr}$ has equilibrium position $\theta_0 \neq 0$, which stability we wont to analyse. Equation of small perturbation α around new equilibrium position is

$$\ddot{\alpha} + (\omega_0^2 + 3h\theta_0^2 + Pl\cos\theta_0)\alpha = 0 . \qquad (7)$$

We are dealing with small oscillations with frequency

$$\omega^2 = \omega_0^2 + (3h + P/2ml)\theta_0^2 , \qquad (8)$$

and conclude that it is system with stable bifurcation.

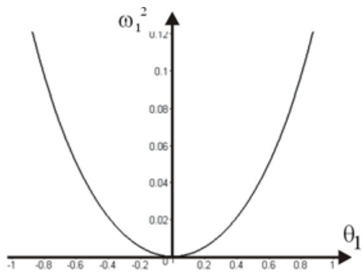

Figure 3. Eigenfrequency of the postcritical equilibrium positions of the system on Fig. 1-b.

Using approximation of deformed shape by buckling mode, lets analyse eigenfrequencies of oscillations of two d.o.f. system around current equilibrium position $\{\theta_1\ \theta_2\}^T$. Supposing motion in shape of harmonic oscillations $\{\alpha_1\ \alpha_2\}^T\sin\omega t$, we get algebraic eigenvalue problem

$$\left(\begin{bmatrix} 2k - Pl & -k \\ -k & k - Pl \end{bmatrix} + \frac{Pl}{6}\theta_1^2 \begin{bmatrix} 1 & 0 \\ 0 & (\theta_2/\theta_1)^2 \end{bmatrix} - ml^2\omega^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \right) \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = 0 . \qquad (9)$$

Varying θ_1 we obtain eigenfrequencies on equilibrium position in the postcritical area (Fig. 3). From eigenfrequency diagram, it is seen that system have stable bifurcation, what also shows analysis of equilibrium path.

5. BIFURCATION OF PHASE TRAJECTORIES

Expressing the equation of motion (1) in terms of velocity $\dot{\theta}$, position θ and initial position θ_0, the equation may be written as

$$ml^2\dot{\theta}^2 + k\theta^2 + h\theta^4/2 + Pl\cos\theta = k\theta_0^2 + h\theta_0^4/2 + Pl\cos\theta_0 \qquad (10)$$

Fig. 4 shows equation (10) in phase space for different value of load and the same initial conditions. Elliptical trajectories around focus (0,0) represents motion for $P < P_{cr}$, while for $P > P_{cr}$ starts symmetrical branching of trajectories to another closed trajectories, what lead to conclusion that system has stable symmetric bifurcation of vertical equilibrium position.

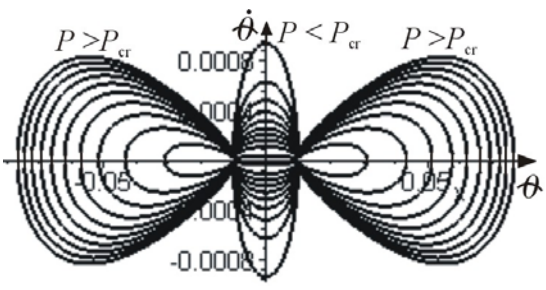

Figure 4. Phase trajectories of the system on Fig. 1- a, given by (10).

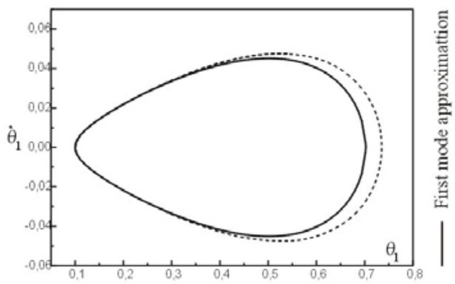

Figure 5. Postcritical phase trajectory of the system on Fig. 1- b.(P = 1.05P_{cr})

In the case of system with two d.o.f. (Fig. 1-b), solving equation of motion using forth order Runge-Kutta method, it is obtained picture of motion in phase space, as shown on Fig. 5. Expressing displacement vector again as $\theta_1\{1\ \theta_2/\theta_1\}^T$, where ratio θ_2/θ_1 corresponds to buckling mode, equation (2) may be written as

$$\dot{\theta}_1^2 = C_1\theta_1^2 + C_2\theta_1^4 - C_1\theta_{10}^2 - C_2\theta_{10}^4 . \qquad (11)$$

where $C_i = C_i(m,l,k,P)$, i = 1, 2, and θ_{10} is initial position.

Phase trajectory identify stable symmetric bifurcation of the equilibrium. Phase trajectories calculated numerically and given by approximative equation (11) are identical in the large part of trajectory (Fig.5). To achieve more accurate results and better approximation of the whole phase trajectory, it is necessary to analyse possibility of approximation of deformed shape using more buckling modes.

6. FREQUENCY CHANGE UNDER LARGE AMPLITUDES

Lets consider oscillations of the system on Fig. 1, taking in account finite values of amplitudes. Oscillations then will be periodic, but not harmonic. Supposing motion in form of Fourier series

$$\theta(t) = \theta_0 + \sum_{i=1}^{n} A_i \cos i\omega t + B_i \cos i\omega t ,$$

and taking in account only the lowest harmonics, $A_1\cos\omega t$ and $B_1\cos\omega t$, we obtain equation for change of the basic frequency of oscillation in the form

$$\omega^2 = \omega_0^2 + P/(8ml)\theta_0^2 \ . \qquad (12)$$

Frequency of large oscillations for different values of load P is presented on Fig. 6, and shows on stable bifurcation.

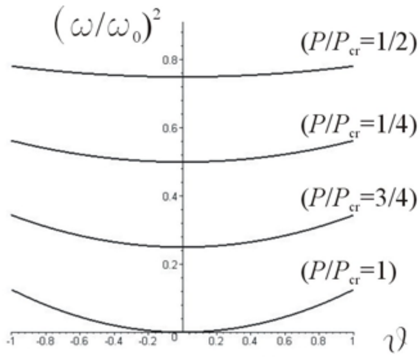

Figure 6. Change of frequency of oscilations caused by amplitude.

7. CONCLUSIONS

In this paper are presented basic methods of linear and nonlinear analysis of stability of elastic systems. Linear analysis can not be used for correct prediction of behaviour of the system in postcritical area. Methods of nonlinear analysis based on the linearization of equilibrium equation could predict it also. Analysis of eigenvalues in linear case is applicable for all type of systems, but its application in postcritical case requires solution of sets of nonlinear equations.

Dynamical approach may be used for its analysis in postcritical case, but it is difficult mathematical problem. Approximation of deformed shape by first buckling mode simplifies this analysis and gives results of acceptable accuracy. Approximation by first buckling mode gives exact shape of phase trajectories, and based on this could be predicted character of stability in postcritical area. Analysis of the frequency change caused by large amplitude around primary equilibrium position in precritical case shows same character of change as eigenfrequencies in postcritical area, and both may be used for analysis of stability after buckling occurred.

It is necessary to analyse applicability of this analysis to systems with different character of bifurcation (unstable symmetric, stable asimmetric, ...), and also develop a method for analysis of systems with more d.o.f. based on the modern numerical methods (Finite Elements Method, Finite Volume Method, ...).

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