THE STUDY OF THE SPRING WITH A CONCENTRATED VERTICAL LOAD IN ITS PLANE BY TRANSFER-MATRIX METHOD

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ABSTRACT
This work presents a study of the analytical calculus for springs, challenged with a concentrated vertical load in its plane, using the Transfer-Matrix Method. We write the basic equations of the spring theory and the Transfer-Matrix for one spring, challenged in its plane. We are deducted the general expression for the Transfer-Matrix of a round spring, challenged in its plane, with an application for a round spring with a concentrated vertical load in its plane, using the Dirac's and Heaviside's distribution functions and operators.

Keywords: state vector, Transfer-Matrix Method, round spring, Dirac's function, Heaviside's function.

1. INTRODUCTION
The study of the springs is very important for a lot of industrie domains. Using the Transfer-Matrix Method, we can write the basic equations of the spring theory with Dirac's and Heaviside's functions and operators and we calculate the six elements of the origin state vector. After, we can calculate, in all spring sections, the state vectors.

2. THE BASIC EQUATIONS OF THE SPRING THEORY
We have studied a circular spring with a constant inertia and have kept the sign conventions - as well as to the beams - for the internal efforts, for the displacements and for the exterior loads. The displacements owing to the cutter force is neglected face to the displacements due by the flexion moment [1]. We have considered a part of a circular spring (Γ), with a mobile reference system (t, n), connected at a current point M (Figure 1, a., [1]).

We have isolated a line element dx (Figure 1., b.) - a curved element ds - challenged by the exterior loads p(x)dx - after Ox axle and q(x)dx - after Oy axle.
We can write for the element, the balance equations and after integration we have:

\[ T_{x}(\theta) = T_{x0} + p_{1}(\theta) \]  

(1)
\[ T_y(\theta) = T_{y(\theta)} - q_1(\theta) \]
\[ M(\theta) = M_0 + T_{x(\theta)} - T_y(\theta) + r(\theta) \]
with: \[ r(\theta) = \int_0^\theta \left[ q_1(\theta) f'(\theta) - p_1(\theta) g'(\theta) \right] d\theta \]

For a constant inertia, after integration, the expressions for \( \omega, u \) and \( v \) are:

\[ \omega = \omega_0 + \frac{1}{EI} \int h d\theta + \frac{1}{EI} \int \varrho dh d\theta + \frac{1}{EI} \int \varrho d\theta \]
\[ u = u_0 - \omega_0 g(\theta) - M_0 + \frac{1}{EI} \int g' h - T_{x(\theta)} \int g' h + T_{y(\theta)} \int g' h - \frac{1}{EI} \int g' h + \frac{1}{EI} \int g' h \]
\[ v = v_0 + \omega_0 f(\theta) + \frac{M_0}{EI} \int f' h + T_{x(\theta)} \int f' h + T_{y(\theta)} \int f' h + \frac{1}{EI} \int f' h + \frac{1}{EI} \int f' h \]

### 3. THE TRANSFER-MATRIX FOR A SPRING, CHALLENGED IN ITS PLANE

For the spring element \( dx \), we can write the expressions (8):

\[ M(\theta) = M_0 + T_{x(\theta)} - T_{y(\theta)} + r(\theta) \]
\[ T_{x(\theta)} = T_{x(\theta)} - p_1(\theta) \]
\[ T_{y(\theta)} = T_{y(\theta)} - q_1(\theta) \]

\[ \omega(\theta) = M_0 + \frac{1}{EI} \int h + T_{x(\theta)} \frac{1}{EI} \int g' h - T_{y(\theta)} \frac{1}{EI} \int f' h + \frac{1}{EI} \int h + \omega_0 + \frac{1}{EI} \int h + \varrho \]
\[ u(\theta) = -M_0 + \frac{1}{EI} \int g' h - T_{x(\theta)} \frac{1}{EI} \int g' h + T_{y(\theta)} \frac{1}{EI} \int g' h - \omega_0 g(\theta) + u_0 + \frac{1}{EI} \int g' h + \varrho \]
\[ v(\theta) = M_0 + \frac{1}{EI} \int f' h + T_{x(\theta)} \frac{1}{EI} \int f' h + T_{y(\theta)} \frac{1}{EI} \int f' h + \omega_0 f(\theta) + v_0 + \frac{1}{EI} \int f' h + \varrho \]

We have for the spring a state vector at the current point \( M, \{U\}_x \), with six elements:

\[ \{U\}_x = \{M(\theta), T_{x(\theta)}, T_{y(\theta)}, \omega(\theta), u(\theta), v(\theta)\} \]

We can write a matrix relation between this state vector, for the section \( x \) and the state vector for the section \( 0 \), \( \{U\}_0 : \{U\}_x = [T] \cdot \{U\}_0 + \{U\}_x \)

where: \([T]\) is the transfer-matrix for the passage between the section \( 0 \) at the section \( x \) and \( \{U\}_x \) is the state vector for the free term at the section \( x \), or, we can write:

\[ \begin{bmatrix} M_0 \\ T_{x(\theta)} \\ T_{y(\theta)} \\ \omega(\theta) \\ u(\theta) \\ v(\theta) \end{bmatrix} = \begin{bmatrix} 1 & g & -f & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{EI} \int h & \frac{1}{EI} \int g' h & -\frac{1}{EI} \int f' h & 1 & 0 & 0 \\ -\frac{1}{EI} \int g' h & -\frac{1}{EI} \int g' h & -\frac{1}{EI} \int f' h & 1 & 0 & 0 \\ -\frac{1}{EI} \int f' h & \frac{1}{EI} \int f' h & \frac{1}{EI} \int f' h & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_0 \\ T_{x(\theta)} \\ T_{y(\theta)} \\ \omega(\theta) \\ u(\theta) \\ v(\theta) \end{bmatrix} + \begin{bmatrix} r \\ p_1 \\ q_1 \\ -\frac{1}{EI} \int h + \varrho \\ -\frac{1}{EI} \int g' h + \varrho \\ -\frac{1}{EI} \int f' h + \varrho \end{bmatrix} \]

The total transfer-matrix for all sections of the spring is obtained if we give at the parameter \( \theta \) the correspondent value for the spring end.

### 4. THE TRANSFER-MATRIX FOR A CIRCULAR SPRING CHALLENGED IN ITS PLANE

We have a circular spring (Figure 1., c.), with a radius \( R \), an angle \( 2\alpha \) and the origin of the axels system is \( O \), same with the spring origin.

We write the parameter equations:

\[ \begin{align*}
& x = f(\theta) = R[\sin \alpha + \sin(\theta - \alpha)] \\
& y = g(\theta) = R[-\cos \alpha + \cos(\theta - \alpha)]
\end{align*} \]

After the calculus and with \( \theta = 2\alpha \), we have the general transfer-matrix for the spring (13):
The vector $\{U_e\}_e$ of the relation (11) will be calculated function of the exterior density loads.

5. THE CIRCULAR SPRING WITH A CONCENTRATED VERTICAL LOAD CHALLENGED IN ITS PLANE

We study an exemple: a circular spring with a concentrated vertical load $F$, challenged in its plane (Figure 1., c.).

We can write for the densities, with Dirac’s and Heaviside’s fonctions and operators:

$$\left\{ \begin{array}{l}
p(x) = 0 \\
q(x) = -F \delta(x - x_0)
\end{array} \right. \quad (14)$$

We change $x$ by $\theta$ and have the expressions (15):

$$\left\{ \begin{array}{l}
p_1(\theta) = 0 \\
q_1(\theta) = q(x) = -FY(\theta - \theta_0) \\
r(\theta) = -FRY(\theta - \theta_0) [\sin(\theta - \alpha) - \sin(\theta_0 - \alpha)]
\end{array} \right. \quad (15)$$

For $0 = 2\alpha$, we have the general matrix expression for the end $O'$:

$$\begin{bmatrix}
1 & 0 & -2R\sin \alpha & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\frac{2R^2}{EI} \sin \alpha & \frac{2R^2}{EI} \cos \alpha \cos \beta & \frac{2R^2}{EI} \sin \alpha & 1 & 0 & 0 \\
\frac{2R^2}{EI} \sin \alpha & \frac{R^4}{EI} (\cos^2 \alpha - \sin^2 \alpha) & \frac{R^4}{EI} (1 + \cos \alpha + \sin \alpha) & 0 & 1 & 0 \\
\frac{2R^2}{EI} \sin \alpha & \frac{R^4}{EI} (1 - \cos \alpha - \sin \alpha) & \frac{R^4}{EI} (\sin \alpha - \cos \alpha) & 2R\sin \alpha & 0 & 1
\end{bmatrix}$$

The vector $\{U_e\}_e$ of the relation (11) will be calculated function of the exterior density loads.

In this moment, we pose the edge conditions: we have the left edge and the right edge embeded (Figure 1., c.).

The conditions at left edge, in the origin, for $\theta = 0$, are:

$$\left\{ \begin{array}{l}
M(2\alpha) = 0 \\
T(2\alpha) = 0 \\
d(2\alpha) = 0 \\
u(2\alpha) = 0
\end{array} \right. \quad (16)$$
\[
\begin{align*}
\omega_0 &= 0 \\
u_0 &= 0 \\
v_0 &= 0 \\
\end{align*}
\] (17)

and at right edge, for \(\theta=2\alpha\), are:
\[
\begin{align*}
\omega(2\alpha) &= 0 \\
u(2\alpha) &= 0 \\
v(2\alpha) &= 0 \\
\end{align*}
\] (18)

With this conditions and for \(\theta_0=45^0\) and \(2\alpha=90^0\), we have for the expression (16):

\[
\begin{bmatrix}
1 & 0 & -R\sqrt{2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & R^2\sqrt{2} & \left(1-\pi\right) & \frac{\pi R^2 \sqrt{2}}{4E1} & 1 & 0 & 0 \\
0 & 0 & \frac{\pi R^2 \sqrt{2}}{4E1} & \frac{R^3 \left(1-\pi\right)}{2E1} & \frac{R^3 \left(1-\pi\right) \sqrt{2}}{2E1} & R\sqrt{2} & 0 & 1 \\
0 & 0 & \frac{R^3 \left(1-\pi\right)}{4E1} & \frac{R^3 \left(1-\pi\right) \sqrt{2}}{4E1} & \frac{R^3 \left(1-\pi\right) \sqrt{2}}{4E1} & \frac{FR^2}{4E1} & 2 & 1-
\end{bmatrix}
\] (19)

We can write a system with 6 equations and 6 variables. After resolution, the results give the state vectors for the origin face O and for the end O’ of the spring.

6. CONCLUSIONS
The Transfer-Matrix Method is very easy to applied for the spring calculus. We can to programm this calculus and with this code, based by the Matrix-Transfer Method, we can calculated rapidly, in the origin section and in the end section of the spring, the 12 elements of the two state vectors. With this results, we can calculated all the state vectors in all sections for the complete spring.

7. REFERENCES