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DISCRETIZATION of ONE-DIMENZIONAL REACTION-DIFFUZION PROBLEM

Dr.sci Enes Duvnjaković
Faculty of Natural Sciences, Mathematics Department, Tuzla, Bosnia and Herzegovina

ABSTRACT
For singularly perturbed selfadjoint one-dimenzional reaction-diffusion problem, using the Galerkin method with the natural choice of test function, a difference scheme on a non-uniform mesh is given, wich is second-order accurate at nodes for the fixed perturbation parameter. A numerical exaple, wich illustrating the theoretical results, is included.
Key words. Reaction-diffusion problem, difference scheme, Galerkin method .

1. INTRODUCTION

We consider the following two-point boundary value problem

$$L_\varepsilon y := \varepsilon^2 y''(x) - p(x)y(x) = f(x), \quad \text{na} \quad (0,1) \tag{1}$$

$$y(0) = 0\,; \quad y(1) = 0\,, \tag{2}$$

where ε is parameter in $(0,1)$, while the functions *p* and *f* are in $C^2[0,1]$ and *p* satisfies

$$p(x) \geq \alpha > 0\,, \qquad x \in [0,1]. \tag{3}$$

Under these hiphoteses L_ε satisfies the maximum principle and problem (1)-(3) has a unique solution . The solution *y* has, in general, a boundary layer of width $O(\varepsilon)$ at both end points of $[0,1]$ (see [3]).

2. DISCRETIZATION OF THE PROBLEM

Let $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$ be a mesh on the interval $[0,1]$. The test functions $\phi_i \ (i = 1,2,\ldots,n-1)$ are taken, satisfying

$$\overline{L}_\varepsilon \phi_i = \varepsilon^2 \phi_i'' - \overline{p}\phi_i = 0\,, \quad on \quad (x_j, x_{j+1}) \quad (j = 0,1,\ldots,n-1) \tag{4}$$

$$\phi_i(x_j) = \delta_{i,j} \quad (j = 0,1,\ldots,n-1), \tag{5}$$

where $\overline{p}(x) := \overline{p}_j = p(x_j + \frac{h_j}{2})\,,\ on\ [\ x_j, x_{j+1}\ )\ (j = 0,1,\ldots,n-1),$

$$h_j = x_{j+1} - x_j \quad (j = 0,1,\ldots,n-1), \quad and \quad \overline{p}(x_n) = \overline{p}_{n-1}.$$

The test functions $\phi_i \ (i = 1,2,\ldots,n-1)$ have the form

$$\phi_i(x) = \begin{cases} u_{i-1}^{II}(x) & ; \quad if \quad x_{i-1} \leq x \leq x_i \\ u_i^{I}(x) & ; \quad if \quad x_i \leq x \leq x_{i+1} \\ 0 & ; \quad otherwise \end{cases} \quad ,$$

where

$$u_i^I(x)=\frac{\sinh(\beta_i(x_{i+1}-x))}{\sinh(\beta_i h_i)}; \qquad u_i^{II}(x)=\frac{\sinh(\beta_i(x-x_i))}{\sinh(\beta_i h_i)},$$

$$\beta_i=\frac{\sqrt{\overline{p}_i}}{\varepsilon}, \qquad h_i=x_{i+1}-x_i, \quad (i=0,1,...,n-1).$$

The approximate solution is then sought in the form $\overline{y}^h(x)=\sum_{k=1}^{n-1} \overline{y}_k^h \ \phi_k(x); \qquad \overline{y}_o^k = \overline{y}_n^k=0$.

Using the proporties of the basis functions ϕ_k $(k=1,2,...,n-1)$, it is evident that $\overline{y}^h(x_k)=\overline{y}_k^h$ $(k=1,2,...,n-1)$. If we define the operator

$$\overline{B}_\varepsilon(w,v):=(-\varepsilon^2 w',v')-(\overline{p}w,v), \quad (w,v\in H_0^1), \tag{6}$$

then we can find the set of constnts $\overline{y}_k^h$ $(k=1,2,...,n-1)$ from the folowing system of linear equations

$$\overline{a}_{i,i-1}^h\ \overline{y}_{i-1}^h+\overline{a}_{i,i}\ \overline{y}_i^h+\overline{a}_{i,i+1}^h\ \overline{y}_{i+1}^h=(\bar{f},\phi_i) \quad (i=1,2,...,n-1) \tag{7}$$

where $\overline{a}_{i,k}=\overline{B}_\varepsilon(\phi_k,\phi_i)$ $(k=i-1, i, i+1)$ and $\bar{f}(x)$ is defined analogously to $\overline{p}(x)$.

Lemma1. *The difference scheme(7) has a unique solution.*

Proof. The matrix of the difference scheme (7) is an M-matrix, and hence invertible. Δ

Let $\tilde{p}(x)$ and $\tilde{f}(x)$ be continuous piecewise linear Lagrangian interpolants of $p(x)$ and $f(x)$ on $[0,1]$, respektively. So, we have

$$\tilde{p}(x)=\tilde{p}_i=a_i x+b_i \quad on \quad [x_i,x_{i+1}],$$
$$\tilde{f}(x)=\tilde{f}_i=c_i x+d_i \quad on \quad [x_i,x_{i+1}],$$

The aproximate solution is then sought in the form

$$\tilde{y}^h(x)=\sum_{k=1}^{n-1} \tilde{y}_k^h \ \phi_k(x); \qquad \tilde{y}_o^k = \tilde{y}_n^k=0, \tag{8}$$

Using the proporties of the basic functions ϕ_k $(k=1,2,...,n-1)$, it is evident $\tilde{y}^h(x_k)=\tilde{y}_k^h$ $(k=1,2,...,n-1)$. If we define the operator

$$\tilde{B}_\varepsilon(w,v):=(-\varepsilon^2 w',v')-(\tilde{p}w,v), \quad (w,v\in H_0^1), \tag{9}$$

then we can find the set of constants $\tilde{y}_k^h$ $(k=1,2,...,n-1)$ from

$$\tilde{B}_\varepsilon(\tilde{y}^h,\phi_i)=(\tilde{f},\phi_i) \quad , \quad for \quad i=1,2,...,n-1 . \tag{10}$$

from (10) we have following variational difference scheme

$$\tilde{a}_{i,i-1}^h\ \tilde{y}_{i-1}^h+\tilde{a}_{i,i}\ \tilde{y}_i^h+\tilde{a}_{i,i+1}^h\ \tilde{y}_{i+1}^h=(\tilde{f},\phi_i) \qquad (i=1,2,...,n-1), \tag{11}$$

where $\tilde{a}_{i,k}=\tilde{B}_\varepsilon(\phi_k,\phi_i)$ $(k=i-1, i, i+1)$. Now, we have

$$\tilde{a}_{i,k}=\overline{B}_\varepsilon(\phi_k,\phi_i)-((\tilde{p}-\overline{p})\phi_k,\phi_i)=\overline{a}_{i,k}-((\tilde{p}-\overline{p})\phi_k,\phi_i) \quad for\ k=i-1,i,i+1. \tag{12}$$

3. NUMERICAL EXPERIMENT

In this section, we present some numerical results which illustrate the results in section 2. The maximum error E_n bettween analytic solution and numerical solution, i.e.
$E_n=\max_{0\le i\le n}|y(x)-\overline{y}^h(x)|$ for scheme (7) and $E_n=\max_{0\le i\le n}|y(x)-\tilde{y}^h(x)|$ for scheme (11).

The order of convergence, defined in the usual way by $Ord = \dfrac{\ln(E_n) - \ln(E_{2n})}{\ln 2}$.

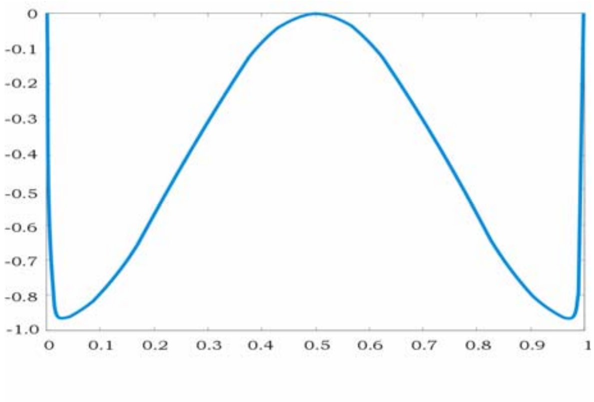
Example. In this example we apply the difference schemes (7) and (11) to problem

$$\varepsilon^2 y'' - y = \cos^2 \pi x + 2\varepsilon^2 \pi^2 \cos 2\pi x$$

$$y(0) = 0 \quad ; \quad y(1) = 0 .$$

The analytic solution of this problem can be writen exsplicitly as $y(x) = \dfrac{e^{-\frac{x}{\varepsilon}} + e^{\frac{x-1}{\varepsilon}}}{1 + e^{-\frac{1}{\varepsilon}}} - \cos^2 \pi x$.

$y(x)$ za $\varepsilon^2 = 2^{-16}$



It can seen from the form of the analytic solution that this problem has a boundary layers in the neighbourhood of points *x=0* and *x=1* for small parameter ε.

In this case we let $h = n^{-1}$; $\varepsilon^2 = 2^{-k}$

Table 1. Scheme (7)

n \ k	3	4	5	6	7	8	10	12	
16	1.86e-2	1.62e-2	1.43e-2	1.33e-2	1.30e-2	1.31e-2	1.38e-2	1.57e-2	E_n
	1.99	1.99	1.99	1.99	2.00	2.02	2.07	2.17	Ord
32	4.68e-3	4.06e-3	3.59e-3	3.33e-3	3.24e-3	3.23e-3	3.28e-3	3.48e-3	E_n
	1.99	1.99	1.99	1.99	2.01	2.02	2.08	2.08	Ord
64	1.17e-3	1.01e-3	8.97e-4	8.32e-4	8.09e-4	8.04e-4	8.07e-4	8.22e-4	E_n
	2.00	2.00	2.00	2.00	2.00	2.00	2.02	2.03	Ord
128	2.92e-4	2.54e-4	2.24e-4	2.08e-4	2.02e-4	2.01e-4	2.01e-4	2.02e-4	E_n
	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.03	Ord
256	7.32e-5	6.35e-5	5.61e-5	5.20e-5	5.05e-5	5.02e-5	5.02e-5	5.02e-5	E_n
	1.99	2.00	2.00	2.00	1.99	1.99	1.99	2.00	Ord
512	1.83e-5	1.58e-5	1.40e-5	1.30e-5	1.26e-5	1.25e-5	1.25e-5	1.25e-5	E_n
	2.00	2.00	2.00	2.00	2.00	1.99	1.99	2.00	Ord
1024	4.56e-6	4.00e-6	3.50e-6	3.24e-6	3.15e-6	3.13e-6	3.13e-6	3.13e-6	E_n

Table 2. Scheme (11)

n \ k	3	4	5	6	7	8	10	12	
16	9.29e-3	8.06e-3	7.11e-3	6.58e-3	6.36e-3	6.27e-3	5.97e-3	5.11e-3	E_n
	1.99	1.99	1.99	1.99	1.98	1.98	1.93	1.77	Ord
32	2.33e-3	2.02e-3	1.79e-3	1.66e-3	1.61e-3	1.59e-3	1.57e-3	1.50e-3	E_n
	1.99	1.99	1.99	2.00	1.99	1.99	1.98	1.93	Ord
64	5.85e-4	5.08e-4	4.48e-4	4.15e-4	4.04e-4	4.01e-4	3.99e-4	3.94e-4	E_n
	2.00	2.00	2.00	2.00	2.00	2.00	2.00	1.98	Ord
128	1.46e-4	1.27e-4	1.12e-4	1.04e-4	1.01e-4	1.00e-4	1.00e-4	1.00e-4	E_n
	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	Ord
256	3.66e-5	3.17e-5	2.80e-5	2.60e-5	2.52e-5	2.51e-5	2.50e-5	2.50e-5	E_n
	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	Ord
512	9.15e-6	7.94e-6	7.01e-6	6.50e-6	6.31e-6	6.27e-6	6.27e-6	6.27e-6	E_n
	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	Ord
1024	2.28e-6	1.98e-6	1.74e-6	1.62e-6	1.58e-6	1.57e-6	1.57e-6	1.57e-6	E_n

Comparing Table 1 with Table 2 one can see that the errors in Table 2 are about two Times smaller then the coresponding errors in Table 1. The numerical results in Tables indicate the uniform convergence of the difference schemes (7) and (11) at the rate $O(h^2)$.

4. REFERENCES

[1] I.P.Boglaev, A Variational Difference Scheme for a Boundary Value Problem with a Small Parameter in the Highest Derivative, U.S.S.R. Comput. Math. And Math. Phys., 21, (1981), 71-81.

[2] E.O'Riordan and M.Stynes, A Uniformly Accurate Finite-element Method for a Singularly -perturbed one Dimensional Reaction-diffusion problem, Math. Comp.,47 (1986),555-570.

[3] A.B.Vasileva and V.F.Butuzov, Asymptotic Exspansion of Singularly Perturbed Equations, Nauka, Moscow, 1973.

[4] A.H.Schatz and L.B.Wahlbin, On The Finite Element Method for Singularly Perturbed Reaction-difusion Problems in Two and One Dimension,Math.Comp., 40 (1983), 47-89.