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POSTBUCKLING ANALYSIS OF RECTANGULAR PLATES USING FINITE ELEMENT METHOD

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ABSTRACT
This paper presents numerical approach to post-buckling analysis of thin rectangular plates loaded in its plane, where deformation of middle plane is neglected. Numerical analysis is performed by finite elements method, dividing plate into mesh of twelve d.o.f. rectangular Kirchhoff finite elements. Outlines of deriving nonlinear equilibrium equation containing third order nonlinearities, arising from large displacement, are presented. Solution of equilibrium equation is obtained using iterative method. Numerical analysis is simplified using results of eigenvalues of linear stability analysis.
Keywords: rectangular plate, finite elements method, postbuckling analysis

1. INTRODUCTION

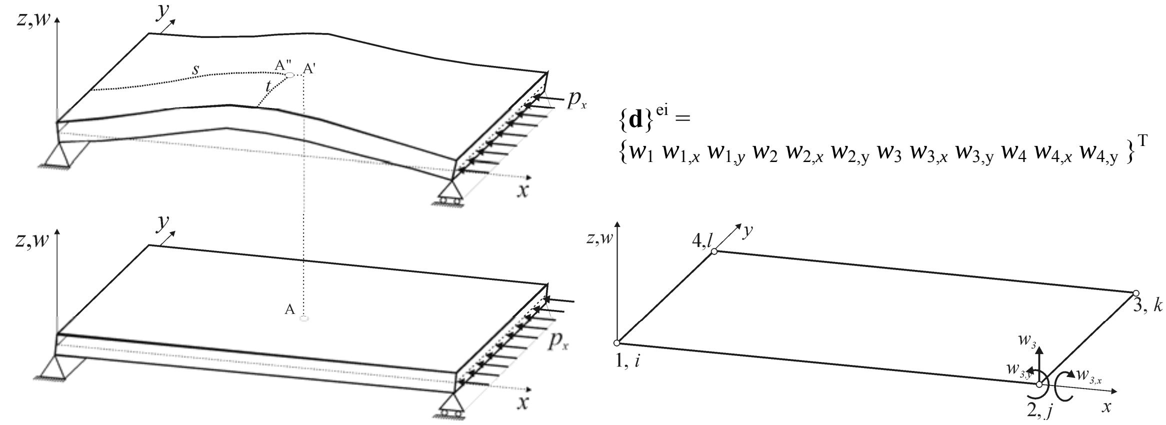
Last decade is characterized by fast development in area of mechanical engineering, expressed through new methods of production techniques, joining methods, new materials and trend of optimum design. But despite of that, plates and beams are still the most used constitutive members in many engineers' constructions. Modern trend of optimisation often turns these members to work near stability boundary. Because of that, their stability analysis is still actual. An analysis of their post-buckling states, which gives answer if construction moves to another equilibrium position or collapses, becomes spatially important.

Some of the first solutions for plate postcritical states are obtained earlier in [1], where results for simply supported plate are calculated by analytical methods, supposing solution (postcritical displacement) in the form of power series. This approach is limited to set of problems with simple boundary condition. Modern approach is based on iterative solution of nonlinear equilibrium equation obtained using some numerical method [3], [4], or on supposing deformation form [5]. But this analysis is leads to very complicated expressions even in case of beam analysis, and in case of plates this fact is naturally more expressed. In presented solutions are shown that, if deformation of plate middle plane exists, plate may bear load several times larger then critical.

This paper presents contribution to finite element approach to postbuckling analysis of rectangular plates loaded in its own plane. It is supposed that supports allow to plate middle plane remains undeformed. Short description of deriving nonlinear equilibrium equation containing third order nonlinearities, which arise from large displacement, is presented. Equation is derived using energy principles, where nonlinearities are included by higher order terms of curvature and plate shortening. Solution of derived equilibrium equation is obtained by using method of linear iterations. This approach is very time consuming, specially if construction of complete postcritical path is required. Stability analysis of beams [6] shows that post-critical analysis may be simplified on the basic of linear stability analysis. The same approach is tested on the problem of stability analysis of the plates, and numerical iterative solution is simplified using results of eigenvalues of linear stability analysis of considered plate.

2. PROBLEM FORMULATION

Postritical analysis of thin rectangular plate is considered. Plate may have different support condition and may have distributed load on all sides or their parts (Fig. 1a). Plate is divided into square Kirchhoff finite elements with four nodes and twelve d.o.f (Fig. 1b).



(a) (b)

Figure 1. (a) Rectangular plate loaded by in-plane load, and its possible postcritical state which is to be determined; (b) Kirchhoff finite element with characteristics degrees of freedom at j-th node.

Large vertical displacement cause change of position of points in *xy* coordinate system. Because of that, configuration of plate is measured in *s* and *t* coordinates. In this case interpolation function is standard polynomial function with twelve unknown constants, which is determined from nodal d.o.f. Assuming derivatives of vertical displacement as sine function of slope, the same shape matrix as in linear analysis of bending, given in [2], may be used.

3. NONLINEAR EQUILIBRIUM EQUATION

Total potential energy of the system is consisted of deformation energy of plate bending and potential energy of applied in-plane load [2], and may be given by the following equation

$$\Pi = (1/2)\sum_{i=1}^{n}\left(\int_0^a\int_0^b\left([\boldsymbol{\kappa}]^{\mathrm{T}}[\mathbf{D}][\boldsymbol{\kappa}]\right)ds dt\right) - P(p_x\varepsilon_s - p_y\varepsilon_t)\,, \tag{1}$$

where $[\boldsymbol{\kappa}]$ is curvature matrix, $[\mathbf{D}]$ is matrix of elasticity constants, ε_s and ε_t are plate shortenings, p_x and p_y are unit distributed loads in *x* and *y* directions, and *P* is load multiplier.

Nonlinear equilibrium equation is derived taking into account higher order terms of curvature

$$\begin{aligned}
\kappa_s &= w(s,t),_{ss} + \frac{1}{2}w(s,t),_{ss}\, w(s,t)^2_{,s}\,,\\
\kappa_t &= w(s,t),_{tt} + \frac{1}{2}w(s,t),_{tt}\, w(s,t)^2_{,t}\,,\\
\kappa_{st} &= 2w(s,t),_{st} + \frac{1}{2}w(s,t),_{st}\, w(s,t)^2_{,s} + \frac{1}{2}w(s,t),_{st}\, w(s,t)^2_{,t}\,,
\end{aligned} \tag{2}$$

and plate shortenings

$$\varepsilon_s = \int_0^a \left(\frac{1}{2}w(s,t)^2_{,s} + \frac{1}{8}w(s,t)^4_{,s}\right)ds\,;\ \ \varepsilon_t = \int_0^b \left(\frac{1}{2}w(s,t)^2_{,t} + \frac{1}{8}w(s,t)^4_{,t}\right)dt\,. \tag{3}$$

Deriving equation for total energy in sense of displacements, equilibrium equation could be obtained in the following matrix form

$$([\mathbf{K}] + P[\mathbf{K}_\sigma])\{\mathbf{D}\} + ([\mathbf{K}_1] + P[\mathbf{K}_{\sigma 1}])\{\mathbf{D}\} = 0\,, \tag{4}$$

where [**K**] is stiffness matrix, [$\mathbf{K}_\sigma$] is stress stiffening matrix, [$\mathbf{K}_1$] is nonlinear stiffness matrix, [$\mathbf{K}_{\sigma 1}$] is nonlinear stress stiffening matrix of the plate and {**D**} is displacement vector.
Nonlinear stiffness and stress stiffening matrices contain coefficients of the second order, and multiplication by displacement vector produce third order nonlinearity in the equilibrium equation.

4. SOLUTION OF EQUILIBRIUM EQUATION

4.1 Direct solution by using linear iterations method

The equilibrium equation (4) is a set of nonlinear algebraic equations, and it is solved directly by using the method of linear iterations. Equation (4) is transformed in the following appropriate form

$$\{\mathbf{D}\} = [\mathbf{K}]^{-1}\left(-[\mathbf{K}_1]\{\mathbf{D}\} + P[\mathbf{K}_\sigma]\{\mathbf{D}\} + P[\mathbf{K}_{\sigma 1}]\{\mathbf{D}\}\right). \tag{5}$$

Considering initial solution $\{\mathbf{D}\}^{(0)}$, solution of equation (4) is determined iteratively by applying equation (5). After solution in *i*-th iteration is calculated, solution of (*i* + 1)-th iteration is then calculated as

$$\{\mathbf{D}\}^{(i+1)} = [\mathbf{K}]^{-1}\left(-[\mathbf{K}_1]\{\mathbf{D}\}^{(i)} + P[\mathbf{K}_\sigma]\{\mathbf{D}\}^{(i)} + P[\mathbf{K}_{\sigma 1}]\{\mathbf{D}\}^{(i)}\right). \tag{6}$$

In the Fig. 2 are shown results of the solutions for the plate continuously simple supported at the all sides and loaded only in one direction (triangles). Solution is expressed through equilibrium path, where dependency of load versus displacement of the central point is plotted. As the initial solutions are used buckling shape $\{\mathbf{D}\}_0$ of the considered plate, calculated from (4), neglecting nonlinear terms and using Power method for extractions of eigenvalues of algebraic eigenvalue problem.

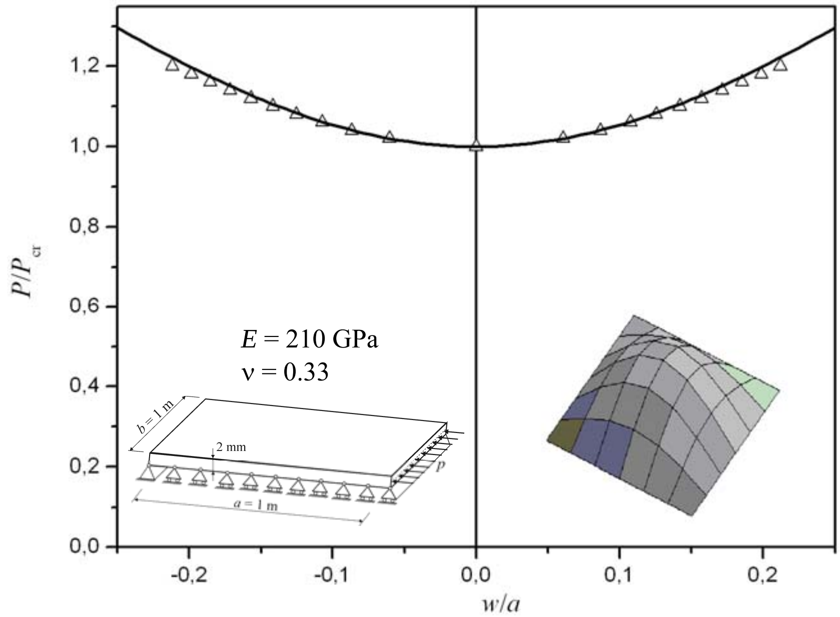


Figure 2. Postcritical equilibrium path for perfect rectangular simple supported plate et all sides: direct solution by using linear iteration (triangles); simplified solution given by (7) (solid line).

In the Fig. 3 is presented dependence between value of applied load and necessary mesh density to achieve convergence of displacement of central point of the plate.

4.2 Simplification of nonlinear analysis

Postbuckling analysis of beams shows that simplification of nonlinear analysis may be done assuming postcritical deformation in shape of buckling mode $\{\mathbf{D}\}_0$ [6]. In this case, dependence of applied load and one characteristic displacement *A* (e.g. the biggest displacement) may be expressed by the following equation

$$P = \frac{\{\bar{\mathbf{D}}\}_0^{\mathrm{T}}[\mathbf{K}]\{\bar{\mathbf{D}}\}_0 + \{\bar{\mathbf{D}}\}_0^{\mathrm{T}}[\mathbf{K}_1]\{\bar{\mathbf{D}}\}_0 A^2}{\{\bar{\mathbf{D}}\}_0^{\mathrm{T}}[\mathbf{K}_\sigma]\{\bar{\mathbf{D}}\}_0 + \{\bar{\mathbf{D}}\}_0^{\mathrm{T}}[\mathbf{K}_{\sigma 1}]\{\bar{\mathbf{D}}\}_0 A^2}. \tag{7}$$

where $\{\bar{\mathbf{D}}\}_0$ is scaled buckling mode $\{\mathbf{D}\}_0$, such that characteristic displacement has unit value.

Simplified solution of the same problem as solved by using method of linear iterations is shown in the Fig. 2 (solid line). The simplified solution predicts same value of the characteristic displacement for $w/a < 0.15$, and after this value of normalized displacement difference increases.

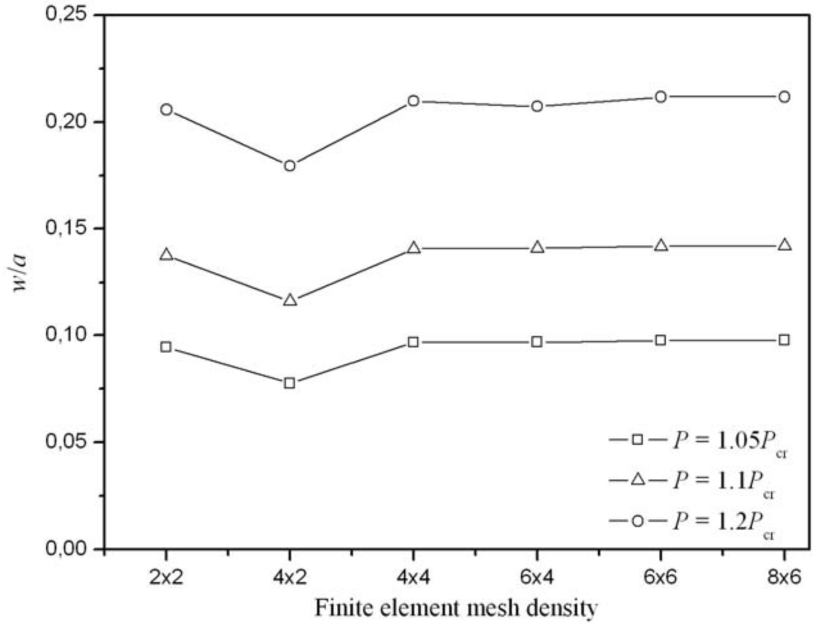


Figure 3. Required mesh density for convergence of displacement of the central point of the plate for intensity of applied load from $1.05P_{cr} \div 1.2P_{cr}$.

5. CONCLUSIONS

This paper presented finite element analysis of postbuckling behaviour of thin rectangular plates, where is supposed that middle plane remains undeformed. Nonlinear equilibrium equation is derived considering higher order terms of curvature and plate shortening. Position of points are measured in plate natural coordinates, what enables usage of same interpolation function as in the linear stability analysis. Obtained equilibrium equation contains third order nonlinear terms, which is stored in constitutive matrices representing nonlinear terms of stiffness and geometrical stiffness of the plate.

Equilibrium equation is solved for simple supported plate using linear iteration method. Obtained results show existence of large vertical displacement of plate in case of undeformed middle plane. It is treated convergence of the results depending on finite element mesh density. It is shown that required mesh density does not depend on intensity of load. Simplification of direct solution is obtained on the basic of linear stability analysis, where postcritical deformation is considered in the shape of buckling eigenmode. According to that, equilibrium equation is turned to single algebraic equation which produce results close to the direct solution of equilibrium equation. This algebraic equation may be used for postcritical analysis instead direct solution of complete equilibrium equation.

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