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LINEAR AND NONLINEAR STABILITY ANALYSIS OF TRUSSES

Vlatko Doleček
Faculty of Mechanical Engineering
Vilsonovo šetalište, Sarajevo
Bosnia and Herzegovina

Safet Isić
Faculty of Mechanical Engineering
University Campus, Mostar
Bosnia and Herzegovina

Avdo Voloder
Faculty of Mechanical Engineering
Vilsonovo šetalište, Sarajevo
Bosnia and Herzegovina

ABSTRACT
This paper presents a comparison of critical loads for trusses, calculated by linear and nonlinear stability analysis. Analyses are provided by using finite elements method. In linear case, critical load is extracted from derived algebraic eigenvalue problem. In case of nonlinear analysis, critical load is determined by construction of postbuckling equilibrium path. Numerical examples for characteristics trusses are given. It is shown that in the case of some perfect trusses, linear approach may produce significant error in the calculation of critical load, and nonlinear analysis should be introduced. The conclusions about conditions for using linear and nonlinear approach to critical load calculation for trusses are derived.
Keywords: truss, stability, linear analysis, nonlinear analysis, critical load

1. INTRODUCTION

Stability analysis of engineering constructions requires calculation of buckling load and corresponding buckling shape. Stability problems are almost simplified by neglecting pre-buckling deformation, and considering construction with no imperfection (perfectly straight beams, etc.). This assumptions enables deriving eigenvalue problem, which solutions are critical (buckling) load and corresponding buckling shape. It is known that in case of beam stability analysis, these assumptions are correct and theoretical value of buckling load, if imperfections are sufficiently small, may be in practice closely obtained [1,2]. In case of beams, axial deformation does not change straight-line state of the beam, but, in case of some perfect trusses, prebuckling deformation may change distribution of forces in constitutive bars, and also acts as imperfection.

Truss-like structures are widely used as load bearing structures, because of their relatively high stiffness related to low mass. In this paper is considered problem of calculation of critical load of truss structure. Both linear and nonlinear calculation is done using finite element method. Linear stability analysis is provided by solving linear algebraic eigenvalue problem, which derivation is also presented. In linear approach, prebuckling deformations are neglected. Nonlinear analysis is performed using linear expressions for constitutive matrices in equilibrium equation. Because of possible large displacements, analysis is done iteratively, checking does equilibrium of forces at every node is satisfied. Residual forces are used as additional nodal forces, until it reaches sufficiently small value. On this approach to nonlinear analysis, prebuckling deformation are taken into account.

Results are compared for the specific two bar truss, commonly used in demonstration of numerical methods [5]. It is shown that in case of trusses, linear approach may lead to large overestimation of critical load, and that control of results using nonlinear analysis should be done.

2. GOVERNING EQUATION

A trusses are structures consisted of pinned straight members (bars), which may only resist axial force (Fig. 1-a, b). One, arbitrary oriented bar is shown in the Fig. 1-c. In the foregoing analysis, bar is modeled by one finite element with two nodes (bar finite element), which have displacement, u and w, along axes of global coordinate system xy, or displacements u' and w' in local coordinate system $x'y'$. Displacements u' and w' of arbitrary point on the element e_i of length l, could be expressed as

$$u'=[\mathbf{N}]\{\mathbf{d}'\}^{e_i}\,;\; w'=[\mathbf{G}]\{\mathbf{d}'\}^{e_i}\,, \tag{1}$$

where shape matrices [**N**] and [**G**], and displacement vector $\{\mathbf{d}'\}^{e_i}$ are given by

$$[\mathbf{N}]=\begin{bmatrix}\frac{x'}{l} & 0 & 1-\frac{x'}{l} & 0\end{bmatrix},\; [\mathbf{G}]=\begin{bmatrix}0 & \frac{x'}{l} & 0 & 1-\frac{x'}{l}\end{bmatrix},\; \{\mathbf{d}'\}^{e_i}=\left\{u_i' \;\; w_i' \;\; u_j' \;\; w_j'\right\}. \tag{2}$$

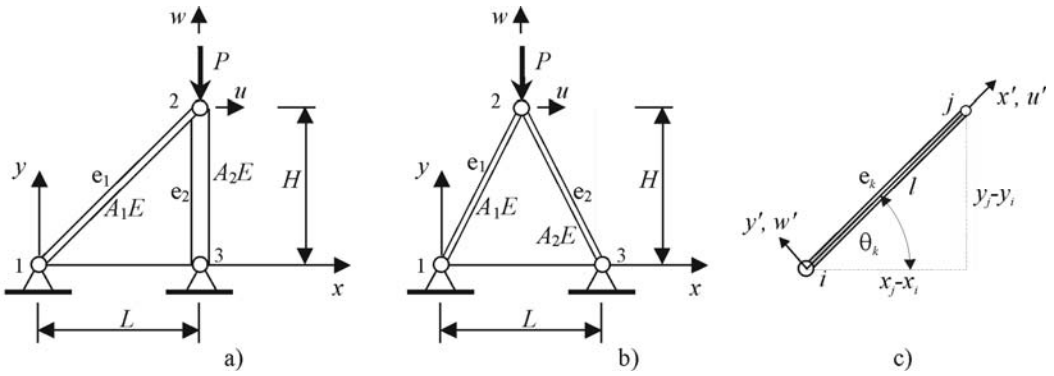


Figure 1. Plane two-bar truss in global coordinate system a), b); bar in local coordinate system (c).

2.1. Linear Stability Analysis

Axial deformation of the single bar in local coordinate system is considered as

$$\varepsilon=\frac{\partial u'}{\partial x'}+\frac{1}{2}\left(\frac{\partial w'}{\partial x'}\right)^2. \tag{3}$$

Using (3), energy of deformation of the complete truss could be written in the following form

$$U=\frac{1}{2}\sum_{i=1}^{n}AEl\left[\left(\{\mathbf{d}'\}^{e_i}\right)^{\mathrm{T}}[\mathbf{N}]^{\mathrm{T}}[\mathbf{N}]\{\mathbf{d}'\}^{e_i}+\frac{1}{2}\left(\left(\{\mathbf{d}'\}^{e_i}\right)^{\mathrm{T}}[\mathbf{G}]^{\mathrm{T}}[\mathbf{G}]\{\mathbf{d}'\}^{e_i}\right)^2\right], \tag{4}$$

where n is total number of bars and A is bar cross sectional area.

Transformation of displacements from local to global coordinate system is done using equations

$$u'=[c\;\; s]\begin{Bmatrix}u_i\\u_j\end{Bmatrix};\; w'=[s\;\; c]\begin{Bmatrix}w_i\\w_j\end{Bmatrix}, \tag{5}$$

where $c = \cos\theta_k$ and $s = \sin\theta_k$.

Using transformations defined by (5), expression for potential energy of the truss may be written in sense of displacements in global coordinate system as

$$U=\frac{1}{2}\sum_{i=1}^{n}AEl\left[\left(\{\mathbf{d}\}^{e_i}\right)^{\mathrm{T}}[\mathbf{k}]^{e_i}\{\mathbf{d}\}^{e_i}+\frac{1}{2}\left(\left(\{\mathbf{d}\}^{e_i}\right)^{\mathrm{T}}[\mathbf{k}_\sigma]^{e_i}\{\mathbf{d}\}^{e_i}\right)^2\right], \tag{6}$$

where $[\mathbf{k}]^{ei}$ is bar stiffness matrix, $[\mathbf{k}_\sigma]^{ei}$ is stress stiffening matrix given by [4]

$$[\mathbf{k}]^{e_i}=\frac{AE}{l}\begin{bmatrix}c^2 & sc & -c^2 & -sc\\ sc & s^2 & -sc & -s^2\\ -c^2 & -sc & c^2 & sc\\ -sc & -s^2 & sc & s^2\end{bmatrix},\; [\mathbf{k}_\sigma]^{e_i}=\frac{1}{l}\begin{bmatrix}s^2 & -sc & -s^2 & sc\\ -sc & c^2 & sc & -c^2\\ -s^2 & sc & s^2 & -sc\\ sc & -c^2 & -sc & c^2\end{bmatrix}. \tag{7}$$

Deriving equation (6) in sense of displacements, equilibrium equation could be written in the form

$$\sum_{i=1}^{n}\left([\mathbf{k}]^{e_i}\{\mathbf{d}\}^{e_i}+S^{e_i}[\mathbf{k}_\sigma]^{e_i}\{\mathbf{d}\}^{e_i}\right)=0, \tag{8}$$

where S^{e_i} is bar axial force, produced by applied external load.
Considering that bar force linearly depends on applied external load *P*, each bar force could be then written as

$$S^{e_i}=\lambda S_0^{e_i}, \tag{9}$$

where $S_0^{e_i}$ is bar force caused by unit external force, and λ is load multiplier.
Using (9), equation (8) may be written in the form of algebraic eigenvalue problem

$$[\mathbf{K}]\{\mathbf{D}\}+\lambda[\mathbf{K}_\sigma]\{\mathbf{D}\}=0, \tag{10}$$

where [**K**]and [$\mathbf{K}_\sigma$] are global stiffness and stress-stiffening matrices, and {**D**} is global displacement vector. Solution of eigenvalue problem (10) are load multiplier λ and corresponding buckling shape.

2.2. Nonlinear Stability Analysis

Considering bar deformation expressed by the linear term of equation (1), equilibrium equation of loaded truss has the known linear form

$$[\mathbf{K}]\{\mathbf{D}\}=\{\mathbf{F}\}, \tag{11}$$

where {**F**} is vector of applied external load (contains one non-zero coefficient in case of trusses in Fig. 1).
If applied load produce large displacement, exact displacements will be different from solution of (11). Coordinates of *i*-th node after deformation are

$$x_i=x_{0i}+D_{ix};\, y_i=y_{0i}+D_{iy}, \tag{12}$$

where x_{0i} and y_{0i} are coordinates of *i*-th node in undeformed state, and D_{ix} and D_{iy} are its displacements calculated from (11).
Real value of axial force in *k*-th bar, which lies between *i*-th and *j*-th node, is then

$$S^{e_k}=\frac{AE}{l_i}\sqrt{(x_j-x_i)^2+(y_j-y_i)^2}. \tag{13}$$

Bar force is, in case of large displacement, different from result of linear analysis. At every node is then calculated components of resultant force ΔF_{ix} and ΔF_{ix} in directions of *x* and *y* axes. Resultant forces in *i*-th node are

$$\Delta F_{ix}=F_{ix}+\sum_{k=1}^{n_{\mathrm{bar}}}S^{e_k}\cos(\theta_k);\ \Delta F_{iy}=F_{iy}+\sum_{k=1}^{n_{\mathrm{bar}}}S^{e_k}\sin(\theta_k), \tag{14}$$

where F_{ix} and F_{iy} are components of external load at *i*-th node, and n_{bar} are number of bars connected in node *i*.
An improved result is obtained by solving equation (11), where matrix [**K**] is calculated for configuration defined by current solutions {**D**}, and load vector {**F**} contains resultant forces, calculated using (14). Calculation is finished when nodes are in equilibrium, i.e. resultant forces become sufficiently small.

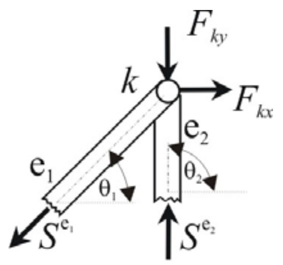


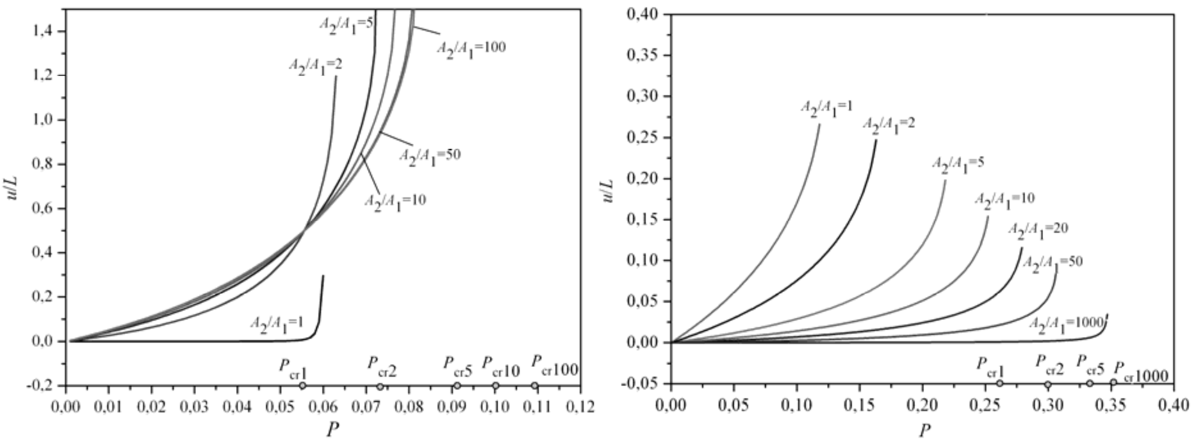
Figure 2. External and internal forces acting on one node.

3. RESULTS DISCUSSION

Results of linear and non-linear calculation of critical load, for characteristic two-bar trusses from Fig. 1, are given in the Fig. 3. Nonlinear calculation of critical load is done by tracking deformation of the system, while applied load increases. Critical load is load at point on the equilibrium path where tangent becomes vertical. Presented procedure of nonlinear analysis may be applied until stable deformed configuration exists.

Results for truss from Fig. 1-a are calculated for $L = H = 1$ m. Truss has unstable symmetric post-critical path [5], and after load P reach critical value no stable deformed shape exists. In case of bars of equal cross-sectional area, calculated critical loads differ more then two times. Critical load calculated by both approaches becomes closed only in case if vertical bar is rigid.

Results for truss from Fig. 1-b are calculated for $L = H/3 = 1$ m. In this case, results of linear and nonlinear analysis become coincident only in case of bars of equal cross-sectional area. Difference increases with ratio of cross-sectional areas of bars, and converges to 27% if one bar is rigid.



(a) (b)

Figure 3. Comparison of postcritical paths.

4. CONCLUSIONS

Presented stability analysis of characteristic trusses shows possibility of large difference of results obtained by linear and non-linear approaches. Error of linear analysis is caused by assumption of force distribution, which neglect deformation of the truss. Results of linear stability analysis are acceptably accurate only if specific geometry (symmetric trusses) or existence of very rigid bars (unsymmetrical trusses) in trusses disables large prebuckling deformations, until critical load predicted by linear analysis is reached.

Analysis shows necessity of usage of non-linear approach to truss stability analysis. For this approach, in this paper is presented efficient and simple method, based on iterative application of linear equilibrium equation.

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