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EIGENFREQUENCY CHANGE OF ELASTIC SYSTEM NEAR CRITICAL POINT

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ABSTRACT
An analysis of eigenfrequencies of characteristic elastic systems near critical point in this paper is presented. It is shown that instead equilibrium path, eigenfrequency change near critical point may be used to determine a nature of bifurcation of equilibrium position.

1. INTRODUCTION
The character of equilibrium stability of conservative elastic system is almost described through dependency of applied load and resulting displacements, i.e. through equilibrium path of a system [2]. Despite the eigenfrequency analysis may be used to detect condition for instability of both conservative and nonconservative system [1, 3], it is not widely used to predict nature of critical points and bifurcation of equilibrium position.
In this paper is presented an analysis of frequency change for different conservative elastic systems, for which is already known nature of bifurcation of equilibrium, determined by static analysis.

2. CONSIDERED ELASTIC SYSTEMS
Analysis is performed on the pendulum-like conservative elastic systems with one d.o.f., presented on Fig. 1. These systems have similar shape and geometry, but exploits different behaviour when applied load is greater then critical. All considered systems consist of attached masses *m*, massles rod of length *l*, and spring of stiffness *k*. Systems on Fig. 1-a is supported by angular spring. System on Fig. 1-b is supported by springs which remains horizontal during deformation of the pendulum, and for system on Fig.1-c inclined spring is used. Differences cause that first system has stable symmetric bifurcation, second has unstable symmetric bifurcation and the third one stable asymmetric.

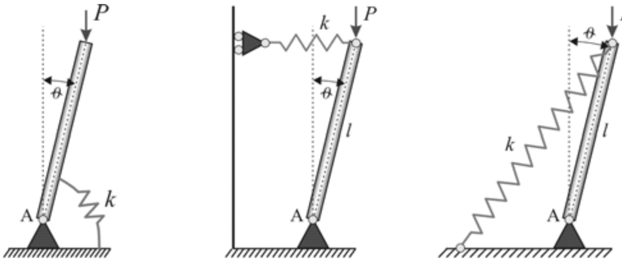

Figure 1. Considered elastic systems.

3. STABLE SYMMETRIC BIFURCATION

Equation of small vibration around vertical equilibrium position of the elastic system on the Fig.1-a is given by

$$ml^2\ddot{\theta}+(k-Pl)\theta=0, \qquad \text{... (1)}$$

what gives the eigenfrequency of vibration

$$\omega=\sqrt{\frac{k-Pl}{ml^2}}. \qquad \text{... (2)}$$

Equation (2) holds for small vibration around vertical position for axial force less than critical. Lets suppose that this elastic system loses stability of vertical position and rotates under action of axial force P. If θ is an angle of new static equilibrium position, small vibration around new position is described by equation

$$\ddot{\alpha}+(\frac{k}{ml^2}-\frac{P}{ml}\cos\theta)\alpha=0, \qquad \text{... (3)}$$

where α is small perturbation around new equilibrium position.
Equation (3) is equation of harmonic vibration and according to it, eigenfrequency of small vibration around new equilibrium position is given by

$$\omega=\sqrt{\frac{k}{ml^2}-\frac{P}{ml}\cos\theta}. \qquad \text{... (4)}$$

Dependency of eigenfrequency on angle θ of equilibrium position is presented on Fig.2. For axial force less then critical eigenfrequency is given by equation (2), what is presented by vertical line. After bifurcation of equilibrium position, two new branches of frequency diagram appears, which are symmetric around ordinate axes and show increasing of the frequency with increasing of equilibrium position angle. It shows that system has stable symmetric bifurcation, as it is depicted by equilibrium path in the static analysis [2].

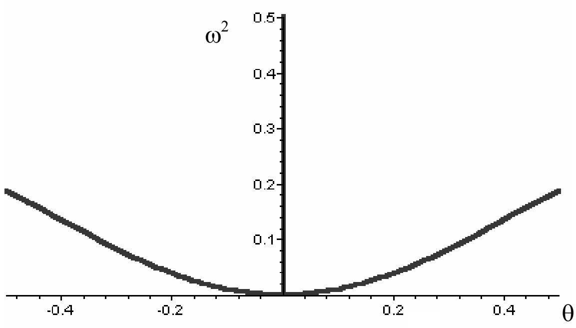

Figure 2. Eigenfrequency change in the case of system with stable symmetric bifurcation.

4. UNSTABLE SYMMETRIC BIFURCATION

In the case of the elastic system on Fig.1-b, equation of small vibration around vertical equilibrium position is given by

$$ml^2\ddot{\theta}+(kl^2-Pl)\theta=0. \qquad \text{... (5)}$$

From equation (5) eigenfrequency of vibration around vertical equilibrium position is

$$\omega = \sqrt{\frac{k}{m} - \frac{P}{ml}}\,. \qquad \text{... (6)}$$

Equation (6) is valid only for small vibration around vertical position when axial force is less than critical. If elastic system loses stability of vertical position and rotates under action of axial force *P*, small vibration around its new position, defined by angle θ, is described by the equation

$$\ddot{\alpha} + \left(\frac{k}{m}\cos 2\theta - \frac{P}{ml}\cos\theta\right)\alpha = 0\,. \qquad \text{... (7)}$$

According to equation (7), eigenfrequency of small vibration around new equilibrium position is then

$$\omega = \sqrt{\frac{k}{m}\cos 2\theta - \frac{P}{ml}\cos\theta}\,. \qquad \text{... (8)}$$

Dependency of eigenfrequency on angle of equilibrium position is presented on Fig.3. For axial force less then critical eigenfrequency is given by equation (6) and presented by vertical line. After bifurcation of equilibrium position two new branches of frequency diagram appears, which are symmetric around ordinate axes but shows decreasing of the frequency with increasing of equilibrium position angle. It shows that system has unstable symmetric bifurcation, as shown by the static analysis [2].

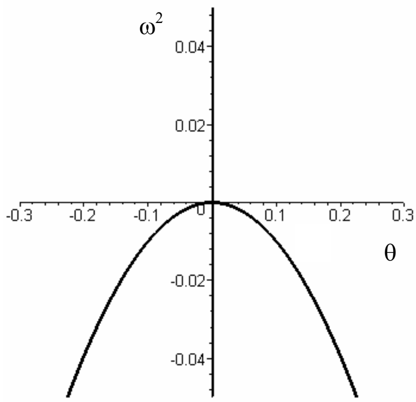

Figure 3. Eigenfrequency change in the case of system with unstable symmetric bifurcation.

5. STABLE ASYMMETRIC BIFURCATION

In the case of the elastic system on Fig.1-c equation of small vibration around vertical equilibrium position is very similar to previously presented system

$$ml^2\ddot{\theta} + \left(\frac{1}{2}kl^2 - Pl\right)\theta = 0\,, \qquad \text{... (9)}$$

where slope angle of the spring to horizontal line of $\pi/4$ is assumed.

From equation (9) eigenfrequency of vibration around vertical equilibrium position is

$$\omega = \sqrt{\frac{k}{2m} - \frac{P}{ml}}\,. \qquad \text{... (10)}$$

Equation (10) is valid only for small vibration around vertical position when axial force is less than critical. If elastic system loses stability of vertical position and rotates under action of axial force *P*, small vibration around its new position defined by angle θ is described by equation

$$\ddot{\alpha}+\frac{\sqrt{2}k}{m}\left[-\frac{1}{2}(\sqrt{1+\sin\theta_o}-1)\cos(\frac{\pi}{4}-\frac{\theta_o}{2})+\frac{1}{2}\frac{\cos\theta_o}{\sqrt{1+\sin\theta_o}}\sin(\frac{\pi}{4}-\frac{\theta_o}{2})-\frac{P}{ml}\cos\theta_o\right]\alpha=0 \quad ...(11)$$

According to equation (11), eigenfrequency of small vibration around new equilibrium position is

$$\omega=\sqrt{\frac{\sqrt{2}k}{m}\left[-\frac{1}{2}(\sqrt{1+\sin\alpha_o}-1)\cos(\frac{\pi}{4}-\frac{\alpha_o}{2})+\frac{1}{2}\frac{\cos\alpha_o}{\sqrt{1+\sin\alpha_o}}\sin(\frac{\pi}{4}-\frac{\alpha_o}{2})\right]-\frac{P}{ml}\cos\alpha_o}\,. \quad ...(12)$$

Dependency of eigenfrequency on angle of equilibrium position is presented on Fig.4. For axial force less then critical eigenfrequency is given by equation (10) is presented by vertical line. After bifurcation of equilibrium position two new branches of frequency diagram appears, which are asymmetric around ordinate axes and show decreasing of the frequency with increasing equilibrium position angle if it is positive and decreasing of the frequency with increasing equilibrium position angle if it is negative. It shows that system has unstable asymmetric bifurcation [2].

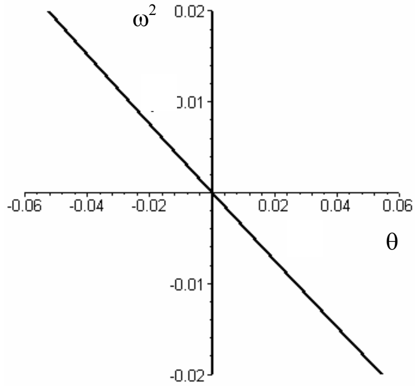

Figure 4. Eigenfrequency change in the case of system with stable asymmetric bifurcation.

6. CONCLUSIONS

The analysis of eigenfrequency change near critical point of conservative elastic systems is presented. Systems with stable symmetric, unstable symmetric and stable asymmetric bifurcation are considered. It is shown that eigenfrequency change near critical point may be used to describe the nature of equilibrium bifurcation of the system.

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