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THE RATIONAL SYSTEM OF EQUATIONS IN THE MODELING ANTI-COMPETITIVE POPULATIONS

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ABSTRACT
In a modelling setting, the anti-competitive system of rational difference equations in the plane

$$x_{n+1} = \frac{y_n}{A_1 + B_1 x_n + C_1 y_n} \quad y_{n+1} = \frac{x_n}{A_2 + B_2 x_n + C_2 y_n}, \quad n = 0,1,2,\dots .$$

System so-called anti-competitive system is know as system (39,39). In a modelling setting system represents evaluation of a system ot two species, where each species in n-th generation helps the growth of the other in (n+1)-th generation (cooperation), while at the same time is decreasing its own size (self-inhibition). In this paper we will investigate the rate of convergence of a solution that convergence to the equilibrium $E_1(0,0)$ of a rational system of difference equations in the plane, where the all parameters are positive numbers, and conditions x_0 and y_0 are arbitrary non-negative numbers.

Key words: difference equations, anti-competitive, global stability, rate of convergence.[1]

1. INTRODUCTION
The system of a difference equations

$$x_{n+1} = \frac{y_n}{A_1 + B_1 x_n + C_1 y_n} \quad y_{n+1} = \frac{x_n}{A_2 + B_2 x_n + C_2 y_n}, \quad n = 0,1,2,\dots . \tag{1}$$

where the parameters A_1, B_1, C_1, A_2, B_2 and C_2 and are positive numbers, and initial conditions x_0 and y_0 are arbitrary nonnegative numbers, has been investigated in [7].

System (1) provides an example of dynamics that is characters for anti-competitive systems. The global dynamics of many subsystems of (1) can be obtained by taking one or more parameters to be 0 in our results. The equilibrium points $(\bar{x}, \bar{y})$ of a system (1) satisfy the system of equations

$$\bar{x} = \frac{\bar{y}}{A_1 + B_1 \bar{x} + C_1 \bar{y}} \quad \bar{y} = \frac{\bar{x}}{A_2 + B_2 \bar{x} + C_2 \bar{y}}. \tag{2}$$

The equilibrium of system (1) are $E_1 = (0,0)$ wich always exists, and $E_2 = (\bar{x}, \bar{y})$ for $\bar{x} \neq 0$, $\bar{y} \neq 0$ from the system (2) can be obtained

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$$\bar{x} = \frac{\bar{y}(A_2 + C_2 \bar{y})}{1 - B_2 \bar{y}}, \ \bar{y} \neq \frac{1}{B_2}, \ \bar{y} = \frac{\bar{x}(A_1 + B_1 \bar{x})}{1 - C_1 \bar{x}}, \ \bar{x} \neq \frac{1}{C_1} \tag{3}$$

These two equilibrium curves intersect in the first quadrant at the points: $E_1 = (0,0)$ and $E_2 = (\bar{x}, \bar{y})$.

Analysis linearized stability indicates several cases with different asymptotic behavior depending on the values of parameters A_1, B_1, C_1, A_2, B_2 and C_2.
The following global asymptotic stability result has been obtained in [7].

Theorem 1.1
a) Assume that $A_1 A_2 > 1$. *Then system (1) has unique equailibrium solution* $E_1 = (0,0)$ *which is globaly asymptotic stable.*
b) Assume that $A_1 A_2 = 1$. *That system (1) has a unique equilibrium solution* $E_1 = (0,0)$ *which is non-hyperbolic and is a globaly attractor.*

Our goal is a to investigate the rate of convergence of solution of a system (1) that converges to the equilibrium $E_1 = (0,0)$ in the regions parameters described in Theorem 1.1. The rate of convergence of solutions that convergence to an equilibrium has been obtanied for some two-dimensional system in [3], [4] and [8]. The following results gives the rate of convergence of solutions of a systema difference equations

$$\mathbf{x}_{n+1} = \left[A + B(n)\right]\mathbf{x}_n , \tag{4}$$

where $\mathbf{x}_n$ is a *k*-dimensional vectors, $A \in C^{k\times k}$ is a constans matrix, and $B : \mathbf{Z}^+ \to C^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \text{ when } n \to \infty , \tag{5}$$

where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm.

Theorem 1. 2. ([5]) *Assume that condition (5) hold. If* $\mathbf{x}_n$ *is a solution of system (4), then either* $\mathbf{x}_n = \mathbf{0}$ *for all large n or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|} \tag{6}$$

exist and is equal to the moduls of one the eigenvalues of matrix A.

Theorem 1. 3. ([5]) *Assume that condition (5) hold. If* $\mathbf{x}_n$ *is a solution of system (4), then either* $\mathbf{x}_n = \mathbf{0}$ *for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|} \tag{7}$$

exist and is equal to the moduls of one the eigenvalues of matrix A.

2. RATE OF CONVERGENCE

In this section we will determinate the rate of convergence of a solution that converges to the equilibrium $E_1 = (0,0)$, in case describe in Theorem 1.1. But, we will prove this generally theorem.

Theorem 2.1 *Assume that a solution* $\{(x_n, y_n)\}$ *of a system (1) converges to the equilibrium* $E = (\bar{x}, \bar{y})$ *and E is globally asymptotically stable. The error vector*

$$\mathbf{e}_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ y_n - \bar{y} \end{pmatrix}$$

of every solution $\mathbf{x}_n \neq \mathbf{0}$ *of (1) satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = \left|\lambda_i\left(J_T(E)\right)\right| \quad \textit{for some } i = 1,2 , \tag{8}$$

and

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = \left|\lambda_i\left(J_T(E)\right)\right| \quad \textit{for some } i = 1,2 \tag{9}$$

where $\left|\lambda_i\left(J_T\left(E\right)\right)\right|$ *is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium* $J_T\left(E\right)$.

Prof. First we will find a system satisfied by the error terms. The error terms are given

$$x_{n+1}-\bar{x}=\frac{y_n}{A_1+B_1x_n+C_1y_n}-\frac{\bar{y}}{A_1+B_1\bar{x}+C_1\bar{y}}=\frac{y_n\left(A_1+B_1\bar{x}+C_1\bar{y}\right)-\bar{y}\left(A_1+B_1x_n+C_1y_n\right)}{\left(A_1+B_1x_n+C_1y_n\right)\left(A_1+B_1\bar{x}+C_1\bar{y}\right)}$$

$$x_{n+1}-\bar{x}=\frac{\left(A_1+B_1\bar{x}\right)\left(y_n-\bar{y}\right)-B_1\bar{y}\left(x_n-\bar{x}\right)}{\left(A_1+B_1x_n+C_1y_n\right)\left(A_1+B_1\bar{x}+C_1\bar{y}\right)}$$

$$x_{n+1}-\bar{x}=\frac{\left(A_1+B_1\bar{x}\right)}{\left(A_1+B_1x_n+C_1y_n\right)\left(A_1+B_1\bar{x}+C_1\bar{y}\right)}\left(y_n-\bar{y}\right)+\frac{-B_1\bar{x}}{\left(A_1+B_1x_n+C_1y_n\right)}\left(x_n-\bar{x}\right),$$

and

$$y_{n+1}-\bar{y}=\frac{x_n}{A_2+B_2x_n+C_2y_n}-\frac{\bar{x}}{A_2+B_2\bar{x}+C_2\bar{y}}=\frac{x_n\left(A_2+B_2\bar{x}+C_2\bar{y}\right)-\bar{x}\left(A_2+B_2x_n+C_2y_n\right)}{\left(A_2+B_2x_n+C_2y_n\right)\left(A_2+B_2\bar{x}+C_2\bar{y}\right)}$$

$$y_{n+1}-\bar{y}=\frac{\left(A_2+C_2\bar{y}\right)\left(x_n-\bar{x}\right)-C_2\bar{x}\left(y_n-\bar{y}\right)}{\left(A_2+B_2x_n+C_2y_n\right)\left(A_2+B_2\bar{x}+C_2\bar{y}\right)}$$

$$y_{n+1}-\bar{y}=\frac{\left(A_2+C_2\bar{y}\right)}{\left(A_2+B_2x_n+C_2y_n\right)\left(A_2+B_2\bar{x}+C_2\bar{y}\right)}\left(x_n-\bar{x}\right)+\frac{-C_2\bar{y}}{\left(A_2+B_2x_n+C_2y_n\right)}\left(y_n-\bar{y}\right).$$

That is

$$\left.\begin{aligned}x_{n+1}-\bar{x}&=\frac{-B_1\bar{x}}{\left(A_1+B_1x_n+C_1y_n\right)}\left(x_n-\bar{x}\right)+\frac{\left(A_1+B_1\bar{x}\right)}{\left(A_1+B_1x_n+C_1y_n\right)\left(A_1+B_1\bar{x}+C_1\bar{y}\right)}\left(y_n-\bar{y}\right),\\ y_{n+1}-\bar{y}&=\frac{\left(A_2+C_2\bar{y}\right)}{\left(A_2+B_2x_n+C_2y_n\right)\left(A_2+B_2\bar{x}+C_2\bar{y}\right)}\left(x_n-\bar{x}\right)+\frac{-C_2\bar{y}}{\left(A_2+B_2x_n+C_2y_n\right)}\left(y_n-\bar{y}\right).\end{aligned}\right\}\quad(10)$$

Set: $e_n^1=x_n-\bar{x}$ and $e_n^2=y_n-\bar{y}$. Then system (10) can be represented as

$$e_{n+1}^1=a_ne_n^1+b_ne_n^2,\quad e_{n+1}^2=d_ne_n^1+c_ne_n^2,$$

where

$$a_n=\frac{-B_1\bar{x}}{A_1+B_1x_n+C_1y_n},\ b_n=\frac{A_1+B_1\bar{x}}{\left(A_1+B_1x_n+C_1y_n\right)\left(A_1+B_1\bar{x}+C_1\bar{y}\right)},$$

$$c_n=\frac{-C_2\bar{y}}{A_2+B_2x_n+C_2y_n},\ d_n=\frac{A_2+C_2\bar{y}}{\left(A_2+B_2x_n+C_2y_n\right)\left(A_2+B_2\bar{x}+C_2\bar{y}\right)}.$$

Taking the limits of a_n,b_n,c_n and d_n, we obtain

$$\lim_{n\to\infty}a_n=\frac{-B_1\bar{x}}{A_1+B_1\bar{x}+C_1\bar{y}},\ \lim_{n\to\infty}b_n=\frac{A_1+B_1\bar{x}}{\left(A_1+B_1\bar{x}+C_1\bar{y}\right)^2},$$

$$\lim_{n\to\infty}c_n=\frac{-C_2\bar{y}}{A_2+B_2\bar{x}+C_2\bar{y}},\ \lim_{n\to\infty}d_n=\frac{A_2+C_2\bar{y}}{\left(A_2+B_2\bar{x}+C_2\bar{y}\right)^2},$$

that is:

$$a_n=\frac{-B_1\bar{x}}{A_1+B_1\bar{x}+C_1\bar{y}}+\alpha_n,\ b_n=\frac{A_1+B_1\bar{x}}{\left(A_1+B_1\bar{x}+C_1\bar{y}\right)^2}+\beta_n,$$

$$c_n=\frac{-C_2\bar{y}}{A_2+B_2\bar{x}+C_2\bar{y}}+\gamma_n,\ d_n=\frac{A_2+C_2\bar{y}}{\left(A_2+B_2\bar{x}+C_2\bar{y}\right)^2}+\delta_n,$$

where $\alpha_n\to 0,\beta_n\to 0,\gamma_n\to 0$ and $\delta_n\to 0$ when $n\to\infty$.

Now we have system of the form (3): $\mathbf{e}_{n+1}=\left[A+B(n)\right]\mathbf{e}_n$.

Thus, the limiting system of error terms can be written as:

$$\begin{pmatrix} e_{n+1}^1 \\ e_{n+1}^2 \end{pmatrix} = \begin{pmatrix} \frac{-B_1\overline{x}}{A_1+B_1\overline{x}+C_1\overline{y}} & \frac{A_1+B_1\overline{x}}{\left(A_1+B_1\overline{x}+C_1\overline{y}\right)^2} \\ \frac{-C_2\overline{y}}{A_2+B_2\overline{x}+C_2\overline{y}} & \frac{A_2+C_2\overline{y}}{\left(A_2+B_2\overline{x}+C_2\overline{y}\right)^2} \end{pmatrix} \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix}.$$

The system is exactly linearized system of (1) evaluted at the equilibrium $E=\left(\overline{x},\overline{y}\right)$. Then Theorems 1.2 and 1.3 imply the result.

If we get $E_1=\left(\overline{x},\overline{y}\right)=\left(0,0\right)$, then we obtain the following result.

Corollary 2.1 *Assume that $a>1$ and $b>1$. Then the equilibrium point $E_1=\left(\overline{x},\overline{y}\right)=\left(0,0\right)$ is a globally asymptotically stable. The error vector of every solution $\mathbf{x}_n \neq \mathbf{0}$ of (1) satisfies both of the following asymptotic relations:*

$$\lim_{n\to\infty} \sqrt[n]{\|\mathbf{e}_n\|} = \lim_{n\to\infty} \sqrt[n]{x_n^2+y_n^2} = \left|\lambda_i\left(J_T\left(E\right)\right)\right| \text{ for some } i=1,2,$$

and

$$\lim_{n\to\infty} \frac{\|\mathbf{e}_{n+1}\|}{\|\mathbf{e}_n\|} = \lim_{n\to\infty} \sqrt{x_{n+1}^2+y_{n+1}^2} = \left|\lambda_i\left(J_T\left(E\right)\right)\right| \text{ for some } i=1,2,$$

where $\left|\lambda_i\left(J_T\left(E\right)\right)\right|$ is equal to the moduls of one the eigenvalues of the Jacobian matrix evaluted at the equilibrium $J_T\left(E\right)$ i.e. $\lambda_i \in \left\{\frac{1}{a},\frac{1}{b}\right\}$.

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