ABSTRACT

In this paper is given bending theory for cylindrical shell subjected to uniformly distributed couple along helix. At some machines for special purposes particularly at machines for snow cleansing, can be found structural elements of thin cylindrical shells on which are welded helicoidally shells. We assume that the displacement due to force are small compared to those due to torque and the helix are simply supported. For curvilinear coordinates on cylindrical surface are introduced angles $\varphi$, $\psi$, so that the coordinate lines are two families or orthogonal helices. Unknown coefficient in the solution of homogeneous linear differential equation with constant coefficients are determined on basis of four boundary conditions. The location of points on cylindrical shell with maximum displacement is found as well as magnitude of this displacements, it is shown that this location is independent on couple. Previous results are compared with those obtained by FEA software code Pro/MECHANICA which has capability of defining uniformly distributed couple along helix. A good accordance between theoretical and numerical results is obtained.

Keywords: shell, helicoidal, plate, cylindrical, strains, force per unit length, couple per unit length

1. INTRODUCTION

With special machines, such as machines for snow cleansing, helicoidal transporters and similar, structural elements of thin cylindrical shell on which helicoidal shell are welded, can be found. Pressure on helicoidal shell is reduced to cylindrical one by cross-section line of medial surfaces, namely by the helix. In case when helicoidal surfaces is loaded by pressure, the resultant is such load on cylinder is of a continually distributed force in the tangent cylinder plane and a continual couple in tangent direction on helicoidal line on cylinder. The force impact on displacement is neglected in this paper, so we consider the bending theory of cylindrical shell loaded by a continual couple along helix.

2. THEORETICAL CALCULATION OF LOADED SHELLS

2.1. Parameters of cylindrical surface within $\varphi$, $\psi$ coordinate system

As the helix is a curve along which boundary conditions are given it is necessary to introduce curvilinear coordinates on cylindrical surfaces, these being different from the ones that are most natural for the cylinder, and they are the angle and distance along the leather. As curvilinear
coordinates on medial shell surfaces two angles are introduced $\theta_1 = \phi$, $\theta_2 = \psi$, so that coordinates are the lines of two helix families. Parameter equation of medium cylindrical shell surface is [1]:

$$
x = R \cos(\phi + \psi), \quad y = R \sin(\phi + \psi), \quad z = k \phi - \frac{R^2}{k} \psi
$$

(1)

Basic vectors are:

$$
A_i = \frac{\partial R}{\partial \phi} = -iR \sin(\phi + \psi) + jR \cos(\phi + \psi) + kk, \quad A_3 = \frac{\partial R}{\partial \psi} = -iR \sin(\phi + \psi) + jR \cos(\phi + \psi) \cdot \frac{R^2}{kk}
$$

(2)

where $R$ is a point position vector on cylinder. A unit vertical vector is:

$$
A_j = \frac{A_j \times A_3}{|A_j \times A_3|} = -i \cos(\phi + \psi) - j \sin(\phi + \psi)
$$

(3)

Co-$\phi$-variant coefficients of the first fundamental surfaces form are:

$$
A_{i1} = A_1 \cdot A_i = R^2 + k^2, \quad A_{i2} = A_2 \cdot A_i = A_1 \cdot A_2 = 0, \quad A_{i3} = A_3 \cdot A_i = \frac{R^2}{k} (R^2 + k^2)
$$

(4)

whereas contra-variant coefficient are:

$$
A^{i1} = \frac{1}{R^2 + k^2}, \quad A^{i2} = A^{i3} = 0, \quad A^{i3} = \frac{k^2}{R^2 (R^2 + k^2)}
$$

(5)

Co-$\psi$-variant coefficient of the second surface form are:

$$
B_{ij} = A_j \frac{\partial A_i}{\partial \psi} = R, \quad B_{12} = B_{13} = B_{23} = R, \quad B_{21} = A_j \frac{\partial A_i}{\partial \psi} = R
$$

(6)

and mixed coefficient are:

$$
B_{11} = B_{22} = \frac{R}{R^2 + k^2}, \quad B_{12} = B_{21} = \frac{k^2}{R^2 (R^2 + k^2)}
$$

(7)

As $B_{12} \neq 0$, coordinate lines are not curve lines, all the relations of shell theory are given in tensor form. All Christoffels symbols of the first and second type equal zero as they are all the coefficients of the first fundamental form of the constant, thus co-variant derivatives are reduced to partial ones.

### 2.2. Deformation values

Only the displacement in direction of $U_3$ normal will be considered, this implying shell bending. As the cylindrical shell is continually loaded along the helix $\psi = 0$ and for the same helix boundary conditions not depending on $\phi$ coordinate are used, we assume that any value (either deformation or statistical one) independent on $\phi$ angle is distant enough from the shell ends. Deformation in tangent plane computed by using formula

$$
e_{\psi \phi} = \frac{1}{2} (u_{\alpha \beta} + u_{\beta \alpha}) - B_{\alpha \beta} u_3
$$

(8)

in our case, the above assumption included, equals

$$
e_{11} = e_{12} = e_{21} = e_{22} = -R_{u_3}
$$

(9)

Expression for curve change of torsion (Eistins convention on additional is applied)

$$
K_{\alpha \phi} = -\bar{K}_{\alpha \phi} = -\bar{K}_{\beta \alpha} = \left[ u_{\alpha \beta \psi} + B_{\alpha \beta} u_3 + B_{\alpha \beta} u_{\alpha \psi} + B_{\phi \beta} u_{\psi \alpha} - B_{\psi \beta} u_{\alpha \psi} \right]
$$

(10)

are reduced, in our case under the above assumption to

$$
\bar{K}_{\alpha \phi} = \bar{K}_{12} = \bar{u}_3, \quad \bar{K}_{22} = \frac{d^2 U_3}{d \psi^2} - u_3
$$

(11)

Deformation values $\bar{\rho}_{(2\phi)}$ are introduced, being computed by formulae

$$
\bar{\rho}_{(2\phi)}(\phi) = \rho_{(2\phi)} + \left( B_{\phi \phi} u_\phi + B_{\phi \alpha} u_{\alpha \phi} \right) / 2, \quad \rho_{(2\phi)} = -\bar{K}_{(2\phi)}.
$$

(12)

In our case $\bar{\rho}_{(2\phi)}$ values are equal:

$$
\bar{\rho}_{(11)} = \bar{\rho}_{(12)} = 0, \quad \bar{\rho}_{(22)} = -\frac{d^2 U_3}{d \psi^2}
$$

(13)

### 2.3. Differential equation of equilibrium

Due to a $U_1, U_2$ displacement hlegation, only one differential equation of equilibrium is used [2]:
\[
\dot{M}^{(\beta\alpha)} + B_{\alpha\beta} \dot{N}^{\alpha\beta} - B_{\alpha\beta} R^\beta M^{(\alpha\beta)} = 0
\]  
(14)

In our case under the introduced assumptions, in its developed form, it is:

\[
d^2 M/ dt^2 - M - 2M - M + R [N + 2N + N] = 0.
\]  
(15)

2.4. Constitutive relations

Relations between statical and deformation values

\[
\dot{N}^{\alpha\beta} = \left[ \dot{\gamma}_{A\beta} A^{\gamma\delta} + (I - \nu) A^{\gamma\delta} \right] e_{\gamma\delta}, \quad \dot{M}^{(\alpha\beta)} = B \left[ \dot{\gamma}_{A\beta} A^{\gamma\delta} + (I - \nu) A^{\gamma\delta} \right] \dot{\gamma}_{\phi\delta},
\]  
in our case are:

\[
\dot{N}^{11} = -C \frac{R^2 + k^2}{R(R^2 + k^2)^2} u_j, \quad \dot{N}^{12} = -C \frac{(I - \nu) k^2}{R(R^2 + k^2)^2} u_j, \quad \dot{N}^{22} = -C \frac{k^2}{R(R^2 + k^2)^2} u_j,
\]  
(17)

\[
\dot{M}^{11} = -B \frac{R^2 + k^2}{R(R^2 + k^2)^2} \frac{d^2 u_j}{d\phi^2}, \quad \dot{M}^{22} = 0, \quad \dot{M}^{22} = -B \frac{k^2}{R(R^2 + k^2)^2} \frac{d^2 u_j}{d\phi^2}.
\]  
(18)

2.5. Differential equation of equilibrium according to \( U_3 \), measurement, boundary conditions and equation solving

Replacing statistical values expressed through \( U_3 \) displacement by on equilibrium equation, we have

\[
d^2 u_j / d\phi^2 - \frac{K^2 + k^2}{K^2} \frac{d^2 u_j}{d\phi^2} + \frac{C R^2 (R^2 + k^2)}{B K^2} u_j = 0
\]  
(19)

This problem of searching for the displacement field of a thin cylinder dried shell loaded with a continual couple along helix has been reduced to the differential equation by \( u_3 \) with corresponding boundary conditions. The solution is being sought in the interval \( \varphi \in [0, \varphi + = 2\pi k / (R^2 + k^2)] \), where \( \varphi^* \) is the helix cross-section \( \varphi = 0 \) with helix \( \varphi = 0 \) along which the couple is distributed. \( M \) action is along the tangent in it collides the sign with the couple \( 2\dot{M}_{22} \). As the connection, between the physical component \( \dot{M}^{(22)} \) and displacement \( M_3 \) is in the form of

\[
M_{22} = -B \frac{R^2 + k^2}{R^2 (R^2 + k^2)} \frac{d^2 u_j}{d\phi^2}
\]  
(20)

helix points \( \varphi = 0 \) are considered to be joint-connected, thus boundary conditions will be for:

\[
\varphi = 0: \quad u_j = 0, \quad \frac{d^2 u_j}{d\phi^2} = -\frac{R^2(R^2 + k^2)}{k^2} \frac{M}{2B}
\]  
(21)

\[
\varphi = \varphi^*: \quad u_j = 0, \quad \frac{d^2 u_j}{d\phi^2} = \frac{R^2(R^2 + k^2)}{k^2} \frac{M}{2B}
\]  
(22)

By introducing designation \( 2p = (R^2 + k^2) / k^2 \) and \( n^2 = (C / B) \cdot R^2 (R^2 + k^2) / k^4 \) for simplicity, the solution of a typical equation that are conjugationally complex ones \( (n^2 p) \) can be written in the form of \( v_1 = s + it, \quad v_2 = s - it, \quad v_3 = s + it, \quad v_4 = -s + it, \) where \( s = \sqrt{\frac{n^2 p}{2}}, \quad t = \sqrt{\frac{n^2 p}{2}} \).

Then the solution of a differential equation has the form of

\[
u_j(\varphi) = e^{\nu_1} (C_1 \cos \nu_1 + C_2 \sin \nu_1) + e^{-\nu_1} (C_3 \cos \nu_1 + C_4 \sin \nu_1)
\]  
(23)

Solving the system of equation obtained from boundary conditions, constants \( C_1, i = 1, 4 \),

\[
C_1 = -\frac{\bar{u}}{4st} \left( \frac{\sin \nu^*}{\cosh \nu^* - \cos \nu^*} \right), \quad C_2 = -\frac{\bar{u}}{4st} \left( \frac{\sh \nu^*}{\cosh \nu^* - \cos \nu^*} \right), \quad C_3 = -C_1, \quad C_4 = C_2 + \frac{\bar{u}}{2st}
\]  
(24)

where \( \bar{u} = R^2 (R^2 + k^2) M / k^2 \).

Thus, differential equation solution is

\[
u_j(\varphi) = -\frac{\bar{u}}{2st} \left[ \frac{\sin \nu^*}{\cosh \nu^* - \cos \nu^*} \sh \nu \cos \nu + \left( \sh \nu - \frac{\sh \nu^*}{\cosh \nu^* - \cos \nu^*} \cosh \nu \right) \sin \nu \right]
\]  
(25)

The solution obtained is valid for enought from the ends of the cylinder being seized the load on the shell is not self-equilibrating, so it is necessary to prevent the movement of the shell as a solid body.
If we equal the first derivative $U_3$ to zero, we obtain an equation

$$tg \psi = (B \cosh \psi - D) / (C \sinh \psi - A)$$

(26)

where

$$A = s \left( \cosh \psi^* - \cos \psi^* \right), \quad B = t \left( \cosh \psi^* - \cos \psi^* \right),$$

$$C = s \sinh \psi^* + t \sin \psi^*, \quad D = t \cosh \psi^* - s \sin \psi^*$$

(27)

out of which, for concrete values $s, t, \psi^*$, by an iterative method the value $\psi_m$ is determined for which the function $U_3(\psi)$ has an extreme value. It is evident that $\psi_m$ is independent on the moment value $M$. For concrete values of cylindrical shell parameters: $R=127$ [mm], $H=140$ [mm], $k = H/2\pi = 22.2817$ [mm], $E=200000$ [MPa], and shell thickness of 2, 3, 4 and 6 [mm], respectively, a position of the displacement value is maximum and the value of that displacement are determined, which can be seen in Table 1.

### Table 1. Displacement values of cylindrical shell

<table>
<thead>
<tr>
<th>$h_c$ [mm]</th>
<th>$\psi_m$ [rad]</th>
<th>$u_3(\psi_m)/M$ [mm/N]</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>2.2583·10^{-2}</td>
<td>0.9004·10^{-5}</td>
</tr>
<tr>
<td>4</td>
<td>1.8284·10^{-2}</td>
<td>2.0137·10^{-5}</td>
</tr>
<tr>
<td>3</td>
<td>1.5840·10^{-2}</td>
<td>3.5803·10^{-5}</td>
</tr>
<tr>
<td>2</td>
<td>1.2941·10^{-2}</td>
<td>8.0623·10^{-5}</td>
</tr>
</tbody>
</table>

3. COMPARISON OF THEORETICAL AND NUMERICAL RESULTS

To compare theoretical and numerical results the programme package Pro/MECHANICA capable of defining a continual couple along helix is used. In Figure 1, results of displacement in radial direction for model of cylindrical shell with the same parameters from the previous chapter and thickness of 3 [mm] are shown. The shell along its helicoidal line is loaded by a continual moment couple $M=1000$ [Nmm/mm] [3,4].

4. CONCLUSION

By solving the homogenous linear differential equation with constant coefficients of displacement in the direction of the normal to the cylinder, a position of points on cylindrical shell in which the value of displacement is maximum and the value of that displacement are determined. It is show that the position of these points is independent on the intensity of a continual couple. Comparing the results with those ones obtained by the programme package for MKE Pro/MECHANICA, an agreement of theoretical and numerical analyses is obtained, introduced assumptions of the efficiency of the calculation methodology of this model of cylindrical shell is proved.

5. REFERENCES