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MODELING THE FLUID FLOW AROUND ANGLED PROFILES USING SUCCESSIVE CONFORMAL MAPPINGS

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ABSTRACT
This paper shows the results of fluid flow around several angled profiles obtained numerically by use of the software developed by the authors. The software algorithm is based on the method of successive conformal mappings for potential flow. The results presented here refer to the profiles of infinite length and with forehead angles of π/2, π/3, π/4 and π/6. The obtained results are sufficiently accurate and close to the theoretical results obtained using the Schwartz-Christoffel transformation.
Key words: plane potential fluid flow, conformal mapping, Jaukowsky transformation

1. INTRODUCTION
The potential theory and conformal mapping is widely used in the physical sciences. The use of conformal mapping in fluid dynamics refers to the analysis of a plane potential fluid flow. Conformal mapping is transformation of points in a complex plane z to the points of a complex plane w by function w(z), which must be unique, continuous and analytic in the area that is mapped. The streamlines of fluid flow around angled profiles with infinite length are found here using the method of successive conformal mapping. This method was initially used in the theory of filtration in civil engineering and most of the credit for its development has a Russian engineer V.S. Kozlov.

2. STREAMLINES OF THE PLANE POTENTIAL FLUID FLOW
The expression for the velocity field of the fluid in a plane potential flow field is:

$$\vec{v} = \text{grad}\varphi = \nabla\varphi v_x = \frac{\partial\varphi(x,y)}{\partial x}\vec{i} + \frac{\partial\varphi(x,y)}{\partial y}\vec{j} = v_x\vec{i} + v_y\vec{j}, \tag{1}$$

where $\varphi(x,y)$ is a scalar function named the potential of flow velocity. The lines which connect all points where $\varphi(x,y)$=const. are called equipotential lines. The velocity in every point of the fluid field is perpendicular to the equipotential line through that point.
In addition to the equipotential lines, another set of lines named streamlines occupy an important place in the fluid dynamics. Streamlines are curves whose tangents at any point coincide with the velocity of the fluid particle at that field point. Differential equation of a streamline is as follows:

$$v_x dy - v_y dx = 0. \tag{2}$$

Assuming that the streamline $\psi = \psi(x,y)$ can be defined as the function that is constant at streamline points, the equation of streamline can be written in the form:

$$\psi(x,y) = \text{const.} \quad \text{or} \quad d\psi = 0. \tag{3}$$

Equipotential lines $\varphi(x,y) = \text{const.}$ and streamlines $\psi(x,y) = \text{const.}$ are everywhere perpendicular to each other and form an orthogonal grid. These functions φ i ψ satisfy Cauchy-Riemann conditions, so by coupling these two functions a new analytic function can be obtained in the form $W = \varphi + i\psi$. This is so called complex potential. The complex function W allows to transform the system of stream and potential lines in the fluid flow physical plane into the orthogonal straight-line mesh in the $\varphi\psi$ - plane.

3. PLANE POTENTIAL FLUID FLOW AROUND ANGLED PROFILES

This paper analyzes the fluid flow around the profile of infinite length with an angled forehead. The fluid flow is infinitely wide, parallel to the axis of symmetry of the profile at infinity and has a constant velocity v_∞. Consideration will include four profiles, whose forehead line make inclination angles of π/2, π/3, π/4 and π/6 with profile axis, Figure 3.

It is well known that the Schwarz-Christoffel transformation gives the analytical solution for fluid flow around profiles with angles $\pi/2$ and $\pi/4$. This analysis will use a method of successive conformal mappings for which an appropriate computer program has been made. To assess the accuracy and performances of the applied method and algorithm, the results for the streamlines obtained using Schwarz-Christoffel transformation for angles $\pi/2$ and $\pi/4$ are presented side by side in Figure 3. For other two values of the angle profile it will be assumed that the results are accurate to the same degree.

3.1. Method of successive conformal mappings

Using the method of successive conformal mappings, the current flow around the profile was mapped to the half plane of complex potential W. This method allows every single connected area to be mapped to the upper half plane with a certain degree of accuracy.

Let's the area is given in the z-plane in the form of a half plane with a cut-out semi-ellipse, Figure 1. Function

$$\tau = \frac{z + \sqrt{z^2 + b^2 - a^2}}{a + b} \tag{4}$$

maps the area in the z-plane with a cut-out semi-ellipse to the area with a cut-off unit semi-circle in the z-plane.

By symmetrical extending of the unit semi-circle into the unit circle in the τ-plane and using the Joukowsky function

$$\zeta^* = \frac{1}{2}\left(\tau + \frac{1}{\tau}\right), \tag{5}$$

the interior of the unit circle in the τ-plane is mapped to the inside of a two-sided notch (-1,+1) along the real axis ξ* of the ζ*- plane. The relation between ζ* and z is as follows:

$$\zeta^* = \frac{z}{a + b} + \frac{b}{z + \sqrt{z^2 + b^2 - a^2}}. \tag{6}$$

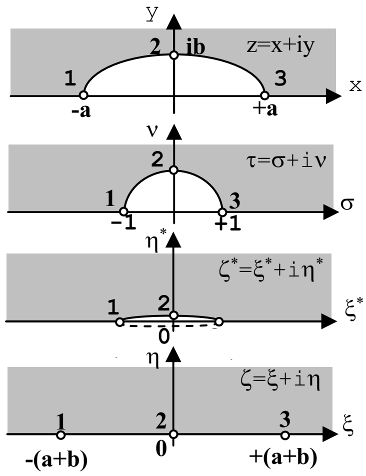

Figure 1. Successive conformal mapping konformnatranslacija KS

New transformation

$$\zeta = (a + b)\zeta^* \tag{7}$$

maps the half-plane ζ* with the notch to the half-plane ζ without the notch, so the transformation function between the z-plane and ζ–plane is in the form:

$$\zeta = z + \frac{b(a + b)}{z + \sqrt{z^2 + b^2 - a^2}} = \frac{az - b\sqrt{z^2 + b^2 - a^2}}{a - b}\ . \tag{8}$$

Let's mark with z_n and z_{n+1} the initial and the final planes after n-th iteration. Then, it can be written:

$$z_{n+1} = z_n^* + \frac{b_n(a_n + b_n)}{z_n^* + \sqrt{(z_n^*)^2 + b_n^2 - a_n^2}} = \frac{a_n z_n^* - b_n\sqrt{(z_n^*)^2 + b_n^2 - a_n^2}}{a_n - b_n}, \tag{9}$$

where is $z_n^* = z_n - b_n$ and a_n, b_n i m_n are parameters of the semi-ellipse that is transformed in the n-th iteration.

The transformation written by Eq.(9) will be marked as $E(m_n, a_n, b_n)$ or E_n. Using the transformation formula (9) it is relatively easy to obtain the coordinates of points after the each iteration. It can be easily proved that after the each iteration the profile contour lowers down and stretches with a tendency to coincide with the real axis.

For any form of profile in the flow plane z, it is necessary to select three points that determine the semi-ellipse whose parameters will determine the next mapping. It is necessary to enable that after the transformation, the boundary area in the new plane does not cross the abscissa, i.e. the point ordinate should not be negative. If A_n, B_n and C_n are the selected points for the n-th mapping, these points will meet the real axis of the z_{n+1} – plane after the transformation, and the ordinates of the other profile points will be reduced. Points located on the abscissa will remain therein after each transformation.
Inverse transformation E_n^{-1} that maps z_{n+1} half-plane to the z_n half-plane with a cut-out semi-ellipse with parameters a_n, b_n and m_n takes place according to the formula:

$$z_n^* = z_n - m_n = \frac{a_n z_{n+1} + b_n \sqrt{z_{n+1}^2 - (a_n + b_n)^2}}{a_n + b_n} \;, \tag{10}$$

which is used in this paper to obtain the streamlines in the physical z-plane using the respective lines in the auxiliary z_{n+1}-plane.

3.2. Using a method of successive conformal mappings to get streamlines around angled profiles

Due to the symmetry of the profile shape, it is enough to analyze the flow field only in one half-plane, for example, in the upper half-plane (above the axis of the profile), Figure 2.

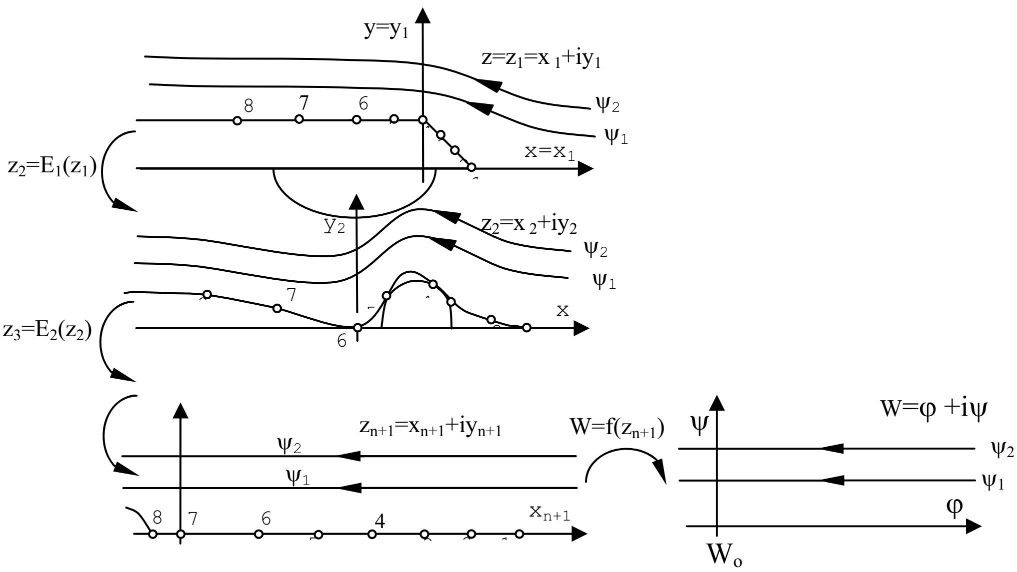

Figure 2. Conformal mappings and complex potential plane

After the sufficient number of mappings E_n in the z_{n+1} - plane it is necessary to specify the value of the ordinate which corresponds to the streamline value in the physical plane z_1. The half-plane z_{n+1} is obtained by a series of conformal mappings with transformation $z_{n+1}= E_1 E_2 ... E_n(z)$, and it represents an auxiliary plane that connects the physical plane z and the complex potential plane W. The relation between the planes z_{n+1} and W is

$$dW / dz_{n+1} = C \,, \quad \text{i.e.} \quad W = Cz_{n+1} + D \,. \tag{11}$$

Provided that point z_o corresponds to the point W_o, the value of D = 0 is obtained. This transformation maps the infinitely distant point into the infinitely distant point too, i.e. the point $z = z_1 \to\infty$ gives $z_2 \to\infty$, and so on up to the $z_{n+1} \to\infty$. Since the speed of the point $z = z_1 \to\infty$ is equal to v_∞, on the basis of complex velocity it can be obtained the value of constant C:

$$\frac{dW}{dz} = \frac{dW}{dz_1} = \frac{dW}{dz_{n+1}} \frac{dz_{n+1}}{dz_n} \frac{dz_n}{dz_{n-1}} \cdots \frac{dz_2}{dz_1} = C\prod_{i=1}^{n} \left[1 - \frac{b_i(a_i + b_i)\left[1 + \dfrac{2(z_i - m_i)}{2\sqrt{(z_i - m_i)^2 + b_i^2 - a_i^2}}\right]}{\left(z_i - m_i + \sqrt{(z_i - m_i)^2 + b_i^2 - a_i^2}\right)^2} \right] = C\prod_{i=1}^{n} P_i \,. \tag{12}$$

This gives

$$\left.\frac{dW}{dz}\right|_{z\to\infty} = v_\infty = C \lim_{z_i\to\infty} \prod_{i=1}^{n} P_i = C\cdot 1 , \tag{13}$$

which results in:

$$W = Cz_{n+1} + D = v_\infty z_{n+1} , \quad \text{i.e.} \quad \varphi + i\psi = v_\infty (x_{n+1} + iy_{n+1}) . \tag{14}$$

Since $\psi = v_\infty y_{n+1}$, the line $y_{n+1} = \psi / v_\infty$ in the final plane corresponds to the streamline of the value ψ. Mapping this line to the plane z=z$_1$ results in a curve that represents the particular streamline. The value of $v_\infty = 1$ is presumed in this analysis.

3.2.1. Description of the procedure for solving the flow around angled profiles

Implementation of successive conformal mappings numerically required making the appropriate computer code. The input data included the coordinates of the points that define the profile and the parameters of the first transformation ellipse. Iterative procedure was implemented until all profile points are lowered to the axis of symmetry (numerically defined condition is $y_i \le 0.01$). The choice of transformation ellipses was based on the largest area. Assigning the streamline value, inverse iterative procedure is performed to return to the initial physical plane. The outputs of the program are the coordinates of the streamlines points in the physical plane and the number of completed iterations.

4. RESULTS FOR STREAMLINES OBTAINED BY THE ANALYSIS

Each of the analyzed profiles as well as a part of the axis of symmetry was defined by a discrete set of 27 to 31 points. The series of 5 iterations for each profile was done.

Figure 3 presents the results obtained numerically for the following values of streamlines ψ: 0.1, 0.2, 0.3, 1, 2 and 3. Figures 3.a. and 3.c present additionally the results theoretically obtained by Schwarz-Christoffel transformation for the forehead angles of π/2 and π/4.

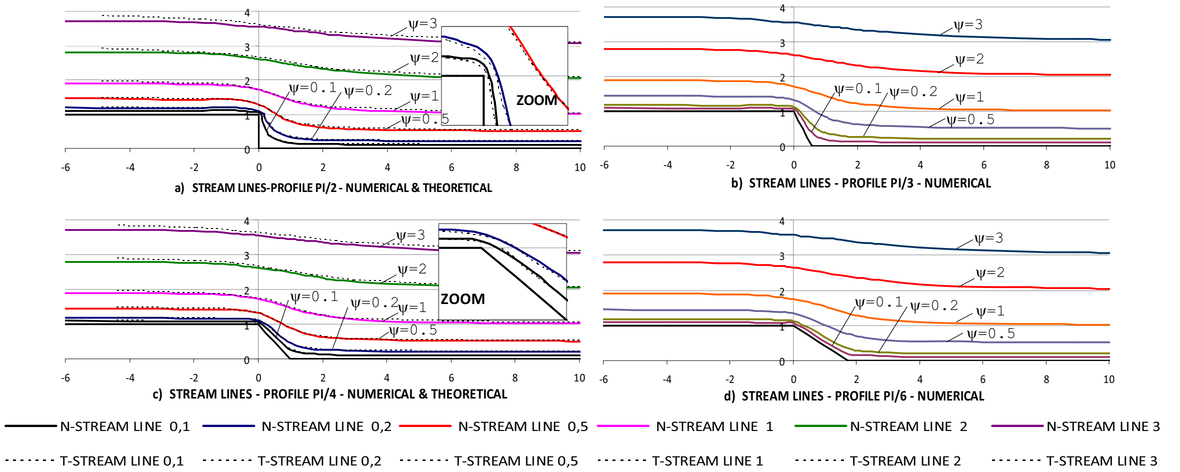

Figure 3. Streamlines for all four analyzed profiles (N-numerical, T-theoretical results)

It can be seen that the streamlines numerically obtained differ very little (of order 10^{-1}) from theoretically obtained results for angles π/2 and π/4. This means that the results for streamlines obtained by the proposed iterative procedure are sufficiently accurate.

5. CONCLUSIONS

Unlike the existing theoretical solutions, the numerical procedure of successive conformal mappings provides a solution for streamlines for any kind of profile that is circulated. The numerical procedure used here gives a solution for any value of the angle profile, which in turn provides the ability to analyze the impact of the angle size on the flow properties. The obtained results are sufficiently close to the theoretical results, which were verified here by comparison with theoretical results obtained by using the Schwartz-Christoffel transformation. For sure, better results would be obtained by a finer profile discretization and by conducting a larger number of iterations.

6. REFERENCES

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