OUT-OF-PLANE DEFLECTION OF J-SHAPED BEAM

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ABSTRACT
In this paper, the problem of calculating deflection of J-shaped beam is presented. Beam is made of two parts: a straight beam and elliptically curved beam. Beam is clamped on curved part and loaded at the end of straight part. Curved part is subjected to both bending and torsion. The Castigliano's theorem is used to determine beam deflection. Numerical integration algorithm in Mathematica software is used for solving integrals that cannot be solved by analytical methods. FEM analysis of curved beam has shown good agreement with analytical solution so it should be further employed for analysis of C-shaped plywood beam which was used as a track on small marble machine.
Keywords: J-shaped beam, out-of-plane deflection

1. INTRODUCTION

During project Making mechanical toys and souvenirs, one of the goals was to make simple marble machine (SMM, Figure 1) as one seen on the site [1] Simple Marble Machine - Busted Bricks.

Figure 1. SMM made form MDF[1]

Instead of MDF board (medium density fiber), material for SMM should be poplar plywood, 4 mm thick. The main concern was the ability of that material to bend. The track had the C shape, but for simplicity, only deflection of the half, J-shaped beam was analyzed analytically and numerically. The similar beam was analyzed in paper [2] where T. Horibe and K. Mori gave solutions for In and Out-of plane deflection, but their beam was clamped on the straight and loaded at the end of curved part. In the paper [3], T. Dahlberg had shown procedure to calculate deflection of the elliptically curved beam. This paper has aim to show detailed analysis of deflection of J-shaped beam which is clamped on the curved end and loaded at the end of straight part.

2. OUT-OF-PLANE DEFLECTION OF J-SHAPED BEAM

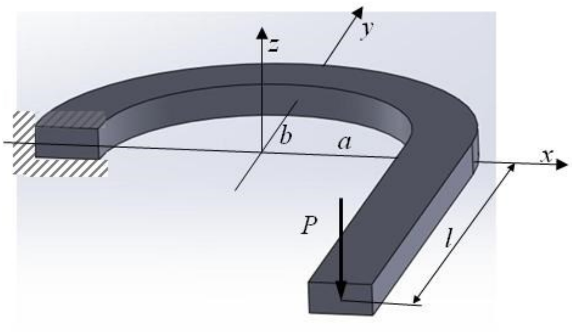

Figure 2. J-shaped beam

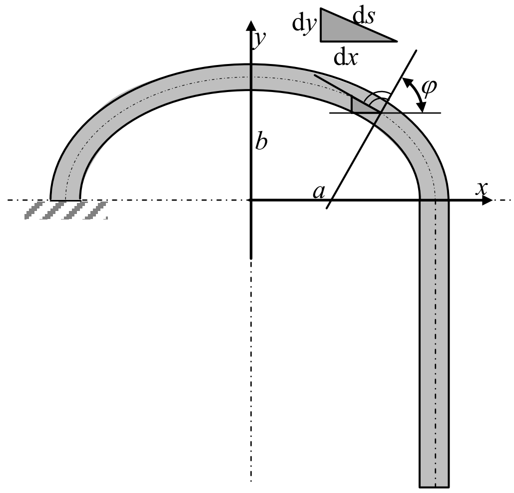

Figure 3 Definition of ellipse geometry

The deflection of the beam end (at the point of application of the force *P* and in the direction of the force, Figure 2) will be determined. The force *P* is normal to the *xy* plane. Half-axes of ellipse are marked as *a* and *b* (Figure 3). Parameter β is introduced, where $b=\beta\cdot a$. The equation of the ellipse is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{1}$$

Solving (1) for *y* gives:

$$y=b\sqrt{1-\frac{x^2}{a^2}}=\beta a\sqrt{1-\frac{x^2}{a^2}} \tag{2}$$

Differentiation of Equation (2) gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\beta x/a}{\sqrt{1-x^2/a^2}} \tag{3}$$

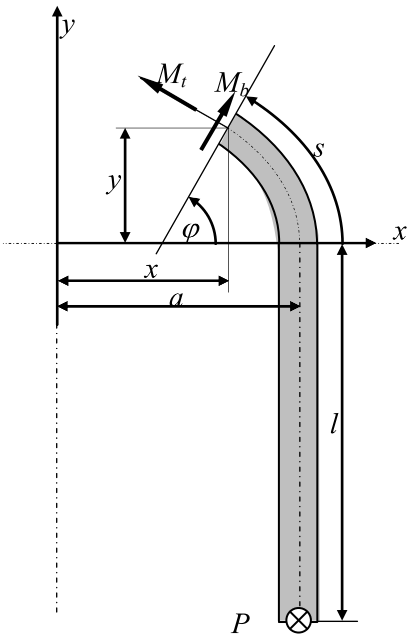

Figure 4 Moments equilibrium

Expression (3) is always negative because d*x* and d*y* have different signs (Figure 3). Here d*x* is negative.
Element d*s* is:

$$\mathrm{d}s=\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2}=-\mathrm{d}x\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \tag{4}$$

The negative root has been selected because d*x* is negative while the length d*s* is positive. Also $\cos\varphi$ and $\sin\varphi$ should be defined:

$$\sin\varphi=-\frac{\mathrm{d}x}{\mathrm{d}s}=\frac{1}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} \qquad \cos\varphi=\frac{\mathrm{d}y}{\mathrm{d}s}=\frac{-\mathrm{d}y/\mathrm{d}x}{\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}} \tag{5}$$

For positive values of *x* and *y* angle φ varies between 0 and $\pi/2$, so both $\cos\varphi$ and $\sin\varphi$ have positive values.

Figure 4 shows cross-section of the elliptical part of beam situated at angle φ. The bending moment *Mb* and torsional moment *Mt* are acting at this cross-section. The shear force has been omitted in the figure, since its influence on beam deflection can be neglected. The equilibrium of moments is used and equations are obtained in the *x* and *y* directions:

$$\Sigma M_x = 0 \qquad \Sigma M_y = 0$$

$$M_b\cos\varphi - M_t\sin\varphi + P(l+y) = 0 \qquad M_b\sin\varphi + M_t\cos\varphi + P(a-x) = 0 \tag{6}$$

Solving for *Mb* and *Mt* gives:

$$M_b = -P\left[(l+y)\cos\varphi + (a-x)\sin\varphi\right] \qquad M_t = P\left[(-a+x)\cos\varphi + (l+y)\sin\varphi\right] \tag{7}$$

Elastic strain energy stored in the beam is:

$$U = \frac{1}{2EI}\int_{-l}^{0}\left[P(l+y)\right]^2 \mathrm{d}y + \frac{1}{2EI}\int_0^L M_b^2 \mathrm{d}s + \frac{1}{2GK_\mathrm{t}}\int_0^L M_t^2 \mathrm{d}s \tag{8}$$

Where: l is the length of straight part of beam, L is the length of the elliptical part of beam, E is modulus of elasticity, G is shear modulus, I is second moment of cross-sectional area, K_t is the cross-sectional factor of torsional rigidity. For isotropic material, the expression for shear modulus is valid: $G=E/[2(1+\nu)]$, where ν is Poisson ratio.

Using Castigliano's theorem, the deflection of the beam end at the load P can be calculated:

$$\delta = \frac{\partial U}{\partial P} = \frac{1}{2EI}\int_{-l}^{0} 2P(l+y)\cdot(l+y)\mathrm{d}y + \frac{1}{2EI}\int_0^L 2M_b\frac{\partial M_b}{\partial P}\mathrm{d}s + \frac{1}{2GK_t}\int_0^L 2M_t\frac{\partial M_t}{\partial P}\mathrm{d}s \tag{9}$$

With M_b, M_t, $\partial M_b/\partial P$ and $\partial M_t/\partial P$ from (7), deflection is:

$$\delta = \frac{Pl^3}{3EI} + \frac{P}{EI}\int_0^L\left[-(l+y)\cos\varphi - (a-x)\sin\varphi\right]^2 \mathrm{d}s + \frac{P}{GK_t}\int_0^L\left[(-a+x)\cos\varphi + (l+y)\sin\varphi\right]^2 \mathrm{d}s \tag{10}$$

Expressions for $\sin\varphi, \cos\varphi, \mathrm{d}s, y, \mathrm{d}y/\mathrm{d}x$ [(2)-(5)] now can be replaced in Equation (10). The integration over ds from 0 to L becomes an integration over dx from a to $-a$. Also, expression $b=\beta\cdot a$ should be put in (10). With another substitution of dimensionless integration variable $t=x/a$ the integrals have limits 1 and -1.

$$\delta = \frac{Pl^3}{3EI} + \\ + \frac{Pa^3}{EI}\int_1^{-1} -\frac{\left\{\frac{l}{a}t\beta + \sqrt{1-t^2}\left[1+t(-1+\beta^2)\right]\right\}^2}{\sqrt{(1-t^2)\left[1+t^2(-1+\beta^2)\right]}}\mathrm{d}t + \frac{Pa^3}{GK_t}\int_1^{-1} -\frac{\left[\frac{l}{a}\sqrt{1-t^2} - (-1+t)\beta\right]^2}{\sqrt{(1-t^2)\left[1+t^2(-1+\beta^2)\right]}}\mathrm{d}t \tag{11}$$

$$\delta = \frac{Pl^3}{3EI} + \frac{Pa^3}{EI}I_1 + \frac{Pa^3}{GK_t}I_2 \tag{12}$$

Where integrals I_1 and I_2 are function of the parameter $\beta=b/a$ and the lengths relationship l/a.

With our SMM, the one track has cross-sectional dimensions (u x v): 8mmx4mm.

The axial moment of inertia is: $I_x = \frac{u\cdot v^3}{12} = \frac{8\cdot 4^3}{12} = 42{,}67\ \mathrm{mm}^4$. The factor of torsional rigidity is: $K_t = c_2 u\cdot v^3 = 0{,}229\cdot 8\cdot 4^3 = 117{,}25\ \mathrm{mm}^4$, where c_2 is coefficient for rectangular bar in torsion, taken from the book [4], Beer, F. P.(2014), Mechanics of Materials.

The length of the straight part of beam is l =55 mm. Half-axes of the ellipse are a=b=23,5 mm.

Material of the beam is poplar plywood, 4 mm thick and with 3 layers of veneer (II+⊥+II). It is orthotropic material with different modulus of elasticity in longitudinal, transversal and radial direction of one veneer. These values have been taken from the paper [5], Brezović, Mladen Vladimir, J and Stjepan, P. (2003).

Deflection of above J shaped beam is obtained for isotropic material, so for the further analytical and numerical analysis, one mean value is calculated for modulus of elasticity and Poisson ratio: E = 3633 MPa, ν= 0,35. Shear modulus is calculated by the expression: $G=E/[2(1+\nu)]$ and has a value: G=1346 MPa. Density of poplar plywood is set to value of 400kg/m^3.

For the parameter β=1 and the relation l/a=55/23,5, the integrals I_1 and I_2 can be calculated using their numerical approximation: I_1 = 10,175 and I_2 =22,678. Since the value of the required displacement δ (22.1 mm) at the end of J beam is known, from the expression (12) the force needed for this deflection can be determined. The value of force is P=7,19 N.

For this value of force, deflection of strait part is 2,57 mm, deflection of the curved part due to bending is 6,12 mm, and due to torsion is 13,4 mm.

3. NUMERICAL ANALYSIS OF OUT-OF-PLANE DEFLECTION OF J-SHAPED BEAM

Same dimensions and physical properties from analytical analysis are applied in the FEM model. The J beam is divided in 11010 quadratic tetrahedral elements. One end is fixed and on the other concentrated force *P*=7,19 N is applied.

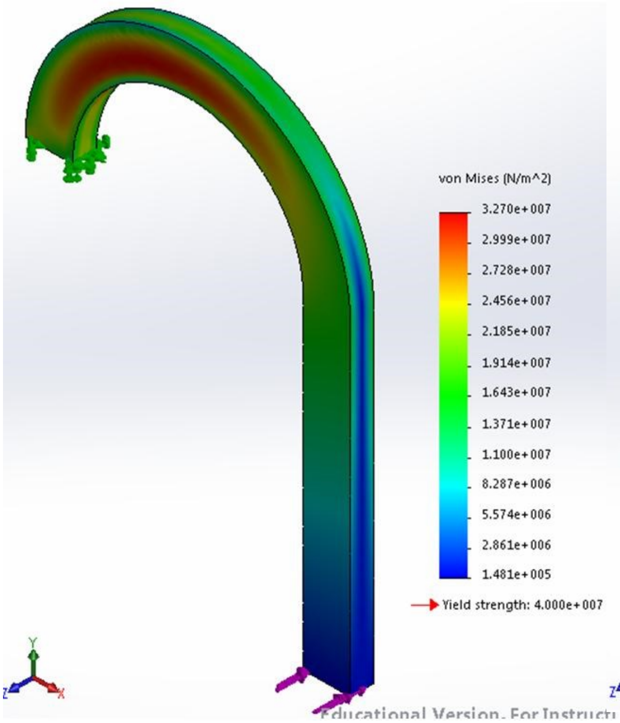

Figure 5 Stress state in J beam

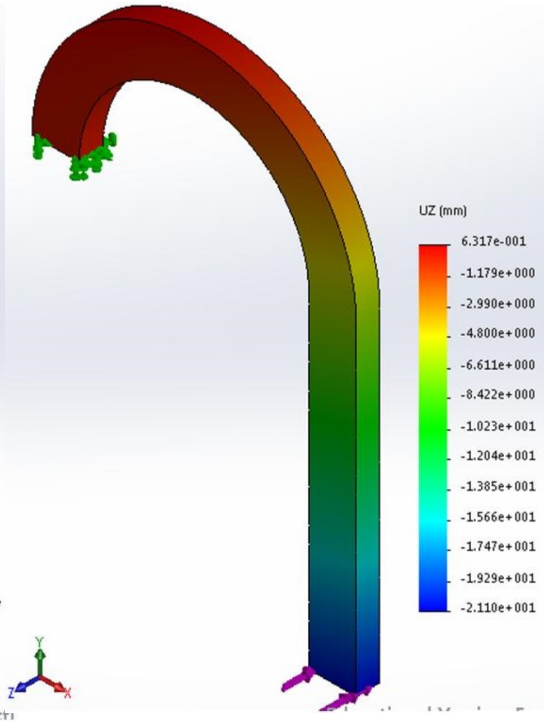

Figure 6 Displacement in z direction

Numerical analysis of the deflection of J beam gives stress and displacement plots (Figure 5 and Figure 6). Maximal value of stress is on the curved part of beam and its value is 32,7 MPa (Yield stress is 40 MPa). Maximal value of displacement in z direction (Figure 6) is in the point of force application and its value is 21,1 mm. This value is close to analytical value of 22,1 mm.

4. CONCLUSION

With confidence in numerical analysis, further analysis of the SMM inner track should be made. This track should be modeled like C shaped beam, made from composite material. It is poplar plywood with three layers of veneers that are glued together with adjacent layers having their wood grain rotated up to 90 degrees to one another. Using numerical analysis optimal shape of beam should be obtained together with possible weak spots in design.

5. REFERENCES

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